Homework 3 Solutions

2-9. Stress components at a point are known, except for the component σ_{yy}

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sigma_{yy} & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

Select σ_{yy} so that there will be a traction-free plane through the point. Determine its unit normal. Known stress components are: $s_{xx} = 1$, $s_{xy} = 0$, $s_{xz} = 1$, $s_{yz} = 2$, $s_{zz} = 0$. We are looking for the components of the unit normal n_x, n_y, n_z and for the stress component s_{yy} such that the components of the traction on the unknown plane $t_x^{(n)}$, $t_y^{(n)}$, $t_z^{(n)}$, be all zero. To this end we use the equations:

$$t_x^{(n)} = \sigma_{xx} n_x + \sigma_{xy} n_y + \sigma_{xz} n_z,$$

$$t_y^{(n)} = \sigma_{xy} n_x + \sigma_{yy} n_y + \sigma_{yz} n_z,$$

$$t_z^{(n)} = \sigma_{xz} n_x + \sigma_{yz} n_y + \sigma_{zz} n_z$$

and obtain:

$$n_x + n_z = 0,$$
$$\sigma_{yy} n_y + 2n_z = 0,$$
$$n_x + 2n_y = 0$$

Thus: $n_z = -n_x$, $n_y = -n_{x/2}$, hence $s_{yy}(-n_{x/2})-2n_x = 0 > (s_{yy}/2+2)n_x=0>$

$s_{yy}=-4$.

Note that n_x can not equal zero as $n_x=0$ would imply $n_y=0$, $n_z=0$ which is impossible because $n_x^2 + n_y^2 + n_z^2 = 1$.
This last equation provides us with sufficient information to find the unit normal's components. : $n^2_x + n^2_x/4 + n^2_x=1$; $n_x=2/3$, $n_y=-1/3$, $n_z=-2/3$ n_x

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2-10. In the beam from problem 2-17 (Fig.P2-10) consider a plane perpendicular to *Oxy* plane, forming a 30° angle with the *0y* axis and passing through the point *M* with the coordinates: *x=L/2, y=0.* Determine the stresses acting on this plane using local Cartesian coordinates *x', y', z'* with the origin at *M* where the plane *Mx'y'* is identical with the plane *Oxy*.
The stresses in the beam are (Problem 2-17):
Hence at point M we have: $s_{xx} = 0$, $s_{yy} = -q/2b$, $s_{xy} = -3qL/4bh$, $s_{xz} = s_{yz} = s_{zz} = 0$.
Now we calculate the direction cosines:

$$\sigma_{xx} = \frac{q}{5bh^3}[20y^3-30y(x^2-\frac{L^2}{4})-3yh^2],$$

$$\sigma_{yy} = \frac{q}{2bh^3}(-4y^3+3yh^2-h^3),$$

$$\sigma_{xy} = \frac{q}{2bh^3}(12xy^2-3xh^2),\ \sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0.$$

2.9

$$n_{x'x} = \cos(x',x) = \cos(2\pi - 30^o) = \frac{\sqrt{3}}{2}, n_{x'y} = \cos(x',y) = \cos(\frac{3\pi}{2} - 30^o) = \frac{1}{2},$$

$$n_{x'z} = \cos(x',z) = 0, n_{y'x} = \cos(y',x) = \cos 60^o = \frac{1}{2}, n_{y'y} = \cos(y',y) =$$

$$= \cos(2\pi - 30^o) = \frac{\sqrt{3}}{2}, n_{y'z} = \cos(y',z) = 0, n_{z'x} = 0, n_{z'y} = 0, n_{z'z} = 1$$

From Eq.(2-35) we obtain

$$\sigma_{x'x'} = \sigma_{yy} n_{x'y}^2 + 2\sigma_{xy} n_{x'x} n_{x'y} = -\frac{q}{4b}(2\frac{1}{4} + 6\frac{L}{h}\frac{\sqrt{3}}{2}\frac{1}{2}) = -\frac{q}{8b}(1 + 3\sqrt{3}\frac{L}{h})$$

From Eq.(2-37) we get

$$\sigma_{x'y'} = \sigma_{yy} n_{x'y} n_{yy'} + \sigma_{xy}(n_{x'y} n_{xy'} + n_{x'x} n_{yy'})$$
$$= -\frac{q}{2b}\frac{1}{2}\frac{\sqrt{3}}{2} - \frac{3qL}{4bh}(\frac{1}{2}\frac{1}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2}) = -\frac{q}{4b}(\frac{\sqrt{3}}{2} + \frac{3L}{h})$$

and from Eq.(2-38)

$$\sigma_{x'z'} = 0$$

Eq.(2-53)$_3$ is identically satisfied. Conclusion: yes, this beam is in equilibrium

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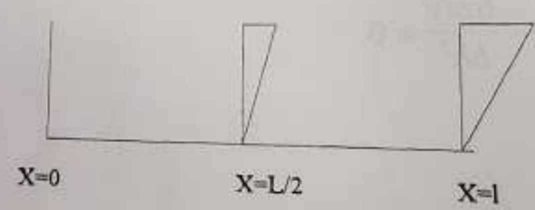
2-16. Given the stress distribution

$$\sigma_{xx} = -\frac{Pxy}{I_z},\ \sigma_{yy} = 0,\ \sigma_{xy} = -\frac{P}{2I_z}\left(\frac{h^2}{4}-y^2\right),\ \sigma_{zz}=\sigma_{xz}=\sigma_{yz}=0$$

in a narrow beam shown in Fig.P2-10. Plot the stress distribution along $x=0, x=L/2, y=0, y=h/2$ and $x=L$. Analyze the equilibrium of the beam and discuss the results.

The plots of stress distribution:

Shear stress distribution



Normal stress distribution

We note that the shear stress is independent of x and that its resultant over any cross section equals P/2. Since the normal stress is a linear function of x, therefore we conclude, by integrating s_{xy} over the cross-section, that the state of stress corresponds to a cantilever beam loaded at the free end x=0 by a force with the resultant P. Now we check the overall equilibrium of the beam. The condition that the sum of all forces acting on beam in the Oy direction must be zero is satisfied identically. The condition for the resultant of all forces acting in the Ox direction becomes:

$$\int_{-h/2}^{+h/2} [\sigma_{xx}(x=0) + \sigma_{xx}(x=L)]dy = -\int_{-h/2}^{+h/2} \frac{PLy}{I_z} dy = 0$$

Also the sum of the moments of all forces acting on the beam with respect to an arbitrary point must be zero. Selecting x=L, y=0 as this point we get:

$$-\int_{-h/2}^{+h/2} \sigma_{xx}(x=L)y\,dy + L\int_{-h/2}^{+h/2} \sigma_{xy}(x=0)dy$$

$$= \frac{PL}{I_z}\int_{-h/2}^{+h/2} y^2 dy + L\int_{-h/2}^{+h/2} [(-\frac{P}{2I_z})(\frac{h^2}{4} - y^2)dy = \frac{PLh^3}{12I_z} - \frac{PL}{2I_z}(\frac{h^3}{4} - \frac{h^3}{12})$$

$$= 0$$

Both equations are identically satisfied.

Section 2-3

2-27. The stress matrix at a point M of the solid is:

$$S = \begin{bmatrix} 1.0 & 0.5 & -1.0 \\ 0.5 & 2.0 & -1.5 \\ -1.0 & -1.5 & -1.0 \end{bmatrix} \times 10^3 \; psi.$$

Find the stress invariants, the principal stresses and the corresponding principal directions

First we identify the stress components from the matrix S:

$$\sigma_{xx} = 1.0, \sigma_{xy} = 0.5, \sigma_{xz} = -1.0, \sigma_{yy} = 2.0, \sigma_{yz} = -1.5,$$
$$\sigma_{zz} = -1.0$$

We calculate the invariants from Eq.(2-68):

2.33

$I_1 = 1.0 + 2.0 - 1.0 = 2.0 \cdot 10^3; I_2 = -0.25 - 2.25 - 1.0 + 2.0 - 2.0 - 1.0 = -4.5 \cdot 10^6;$
$I_3 = 1 \cdot (-2 - 2.25) - .5 \cdot (-0.5 - 1.5) - 1 \cdot (-0.75 + 2) = -4.5 \cdot 10^9$

The characteristic Eq.(2-67) becomes:

$$\sigma_n^3 - 2 \cdot 10^3 \sigma_n^2 - 4.5 \cdot 10^6 \sigma_n + 4.5 \cdot 10^9 = 0$$

Solving this equation we get the eigenvalues:

$$\sigma_1 \approx 3.000 \cdot 10^3\, psi, \sigma_2 \approx 0.823 \cdot 10^3\, psi, \sigma_3 \approx -1.823 \cdot 10^3\, psi,$$

Substituting $\sigma_n = \sigma_1$ into the first and into the second of Eqs.(2-64) we get:

$$-2n_x + 0.5n_y - n_z = 0, 0.5n_x - n_y - 1.5n_z = 0$$

Eq.(2-65) is the third equation defining the direction cosines

$$n_x^2 + n_y^2 + n_z^2 = 1$$

Solving this system of equations we get:

$$n_x^{(1)} = 0.408,\; n_y^{(1)} = 0.816,\; n_z^{(1)} = -0.408$$

Substituting $\sigma_n = \sigma_2$ into the first and into the second of Eqs.(2-64) we get:

$$0.177n_x + 0.5n_y - n_z = 0,\; 0.5n_x + 1.177n_y - 1.5n_z = 0$$

and the third equation is the same as previously

$$n_x^2 + n_y^2 + n_z^2 = 1$$

Solving this system of equations we get:

$$n_x^{(2)} = -0.873, n_y^{(2)} = 0.480, n_z^{(2)} = 0.085$$

Substituting $\sigma_n = \sigma_3$ into the first and into the second of Eqs.(2-64) we get:

$$2.823n_x + 0.5n_y - n_z = 0, 0.5n_x + 3.823n_y - 1.5n_z = 0$$

and again

$$n_x^2 + n_y^2 + n_z^2 = 1$$

Solving this system of equations we get:

$$n_x^{(3)} = 0.265, n_y^{(3)} = 0.322, n_z^{(3)} = 0.909$$

We can verify the results knowing that any two principal directions are mutually orthogonal
is that the dot product of two vectors directed along two principal directions must be zero.
Taking for instance directions "1" and "2" we should have:

$$n_x^{(1)} n_x^{(2)} + n_y^{(1)} n_y^{(2)} + n_z^{(1)} n_z^{(2)} = 0$$

Substituting the numerical results just obtained we get zero. The same result we obtain when taking the directions "2" and "3".

2-28. The stress matrix is

$$S = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 2 & 1 & 4 \end{bmatrix}$$

(a) Determine the principal stresses,
(b) find the direction cosines of the normal to the plane on which acts σ_{max} .
Here:

$$\sigma_{xx} = 1, \sigma_{xy} = 0, \sigma_{xz} = 2, \sigma_{yy} = 3, \sigma_{yz} = 1, \sigma_{zz} = 4$$

(a) First we find the invariants using Eq.(2-68):

$$I_1 = 1 + 3 + 4 = 8, I_2 = -4 - 1 + 3 + 4 + 12 = 14,$$
$$I_3 = 1 \cdot (12 - 1) + 2 \cdot (-3) = 5$$

Hence the characteristic equation (2-67) becomes

$$\sigma_n^3 - 8\sigma_n^2 + 14\sigma_n - 5 = 0$$

Its roots are the principal stresses:

$$\sigma_1 \approx 5.695, \sigma_2 \approx 1.823, \sigma_3 \approx 0.482$$

(b) The maximum stress is σ_1 so we should find the direction cosines corresponding to its value. Substituting $\sigma_n = \sigma_1$ into the first and into the second of Eqs.(2-64) we get:

$$-4.695n_x^{(1)} + 2n_z^{(1)} = 0, -2.965n_y^{(1)} + n_z^{(1)} = 0$$

The third equation we use is Eq.(2-65)

$$n_x^2 + n_y^2 + n_z^2 = 1$$

Solving those three equations we find the direction cosines of the normal to the plane on which acts σ_{max}

$$n_x^{(1)} = 0.374, n_y^{(1)} = 0.296, n_z^{(1)} = 0.879$$