Homework 5 Solutions

#3-11 a). Assume that the x axis in figure P3-11 is the principal axis for ϵ_1 and the y axis is the principal axis for ϵ_2, with $\epsilon_2 > \epsilon_1$. Then from the first of Eqs(3-30), for a strain gage with no mounting error (i.e. $\beta = 0°$):

$$\epsilon_{x'x'} = \frac{\epsilon_1 + \epsilon_2}{2} + \frac{\epsilon_1 - \epsilon_2}{2} \cos(2\theta)$$

whereas with a mounting error of β,

$$\tilde{\epsilon}_{x'x'} = \frac{\epsilon_1 + \epsilon_2}{2} + \frac{\epsilon_1 - \epsilon_2}{2} \cos[2(\theta + \beta)]$$

$\therefore$ the error, E, is

$$E = \tilde{\epsilon}_{x'x'} - \epsilon_{x'x'}$$

$$E = \frac{\epsilon_1 - \epsilon_2}{2} \{\cos[2(\theta+\beta)] - \cos(2\theta)\}$$

But $\cos\alpha - \cos\gamma = -2\sin\left(\frac{\alpha+\gamma}{2}\right)\sin\left(\frac{\alpha-\gamma}{2}\right)$

$\therefore \cos[2(\theta+\beta)] - \cos(2\theta) = -2\sin(2\theta+\beta)\sin\beta$

so that E can be written as

(1) $$E = -(\epsilon_1 - \epsilon_2)\sin\beta \sin(2\theta+\beta)$$

The percent error is

$$E' = \frac{\tilde{\epsilon}_{x'x'} - \epsilon_{x'x'}}{\epsilon_{x'x'}} 100$$

$$E' = \frac{(\epsilon_1 - \epsilon_2)\{\cos[2(\theta+\beta)] - \cos(2\theta)\}}{(\epsilon_1 + \epsilon_2) + (\epsilon_1 - \epsilon_2)\cos(2\theta)} 100$$

(2) $$E' = \frac{\cos[2(\theta+\beta)] - \cos(2\theta)}{\frac{r+1}{r-1} + \cos(2\theta)} 100$$

$$r \equiv \epsilon_1/\epsilon_2$$

Note that $\cos[2(\theta+\beta)] = \cos(2\theta)\cos(2\beta) - \sin(2\theta)\sin(2\beta)$

and ∴ for small values of β,

$$\cos[2(\theta+\beta)] = \cos(2\theta) - 2\beta\sin(2\theta)$$

Thus E and E' become,

(3) $$E = -(\epsilon_1 - \epsilon_2)\beta\sin(2\theta)$$

(4) $$E' = \frac{-2\beta\sin(2\theta)}{\frac{r+1}{r-1} + \cos(2\theta)} 100$$

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b). β, θ, ϵ_1/ϵ_2

c). Consider values of θ in the approximate range of $0° \leq \theta \leq 90°$. The strain given in Eq(1) takes on a minimum magnitude when $\sin(2\theta+\beta)=0$ and a maximum magnitude when $\sin(2\theta+\beta)=1$

$\therefore$ E_{min} occurs at $\theta = -\frac{\beta}{2}$ or $\theta = -\frac{\beta}{2} + \frac{\pi}{2}$

E_{max} occurs at $\theta = -\frac{\beta}{2} + \frac{\pi}{4}$

Thus the strain gage orientations minimum and maximum error differ by 45°.

Note that as $\beta \to 0$ (ie, mounting error $\to 0$)

E_{min} occurs at $\theta \to 0$ or $\theta \to \frac{\pi}{2}$

E_{max} occurs at $\theta \to \frac{\pi}{4}$

In other words, the strain gage reading is least sensitive to mounting errors when mounted along one of the principal axes and most sensitive when mounted at an angle of 45° relative to the principal axes. This can also be readily seen from Eq (3). Note that if $\epsilon_1 = \epsilon_2$ then $E = 0$ for any θ and any β.

Problem 3-11 part d

The error is given by Eq. (1) as

$$E = -(\epsilon_1 - \epsilon_2)\cdot\sin(\beta)\cdot\sin(2\cdot\theta + \beta)$$

For uniaxial strain state,

$$\epsilon_2 = -\nu\cdot\epsilon_1 = -\nu\cdot\epsilon$$

Therefore,

$$E = -\epsilon\cdot(1+\nu)\cdot\sin(\beta)\cdot\sin(2\cdot\theta + \beta)$$

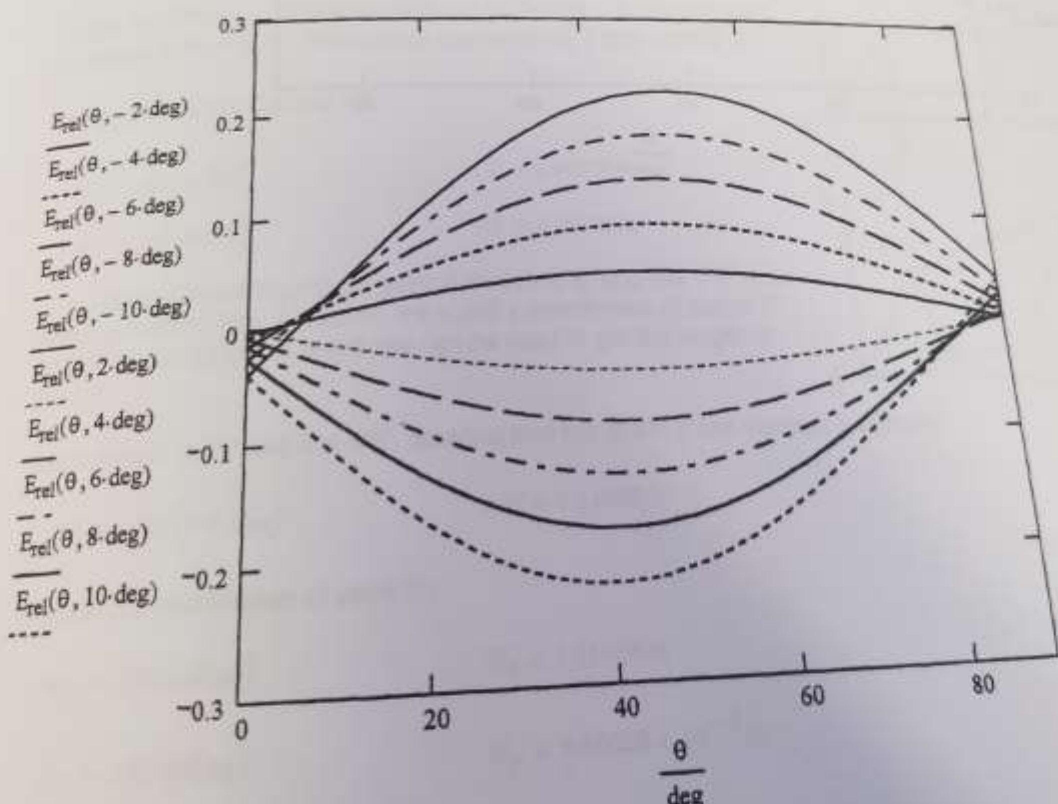
and

$$E_{rel} = \frac{E}{\epsilon} = \frac{(1+\nu)}{2}\cdot\sin(\beta)\cdot\sin(2\cdot\theta + \beta)$$

With $\nu := .285$

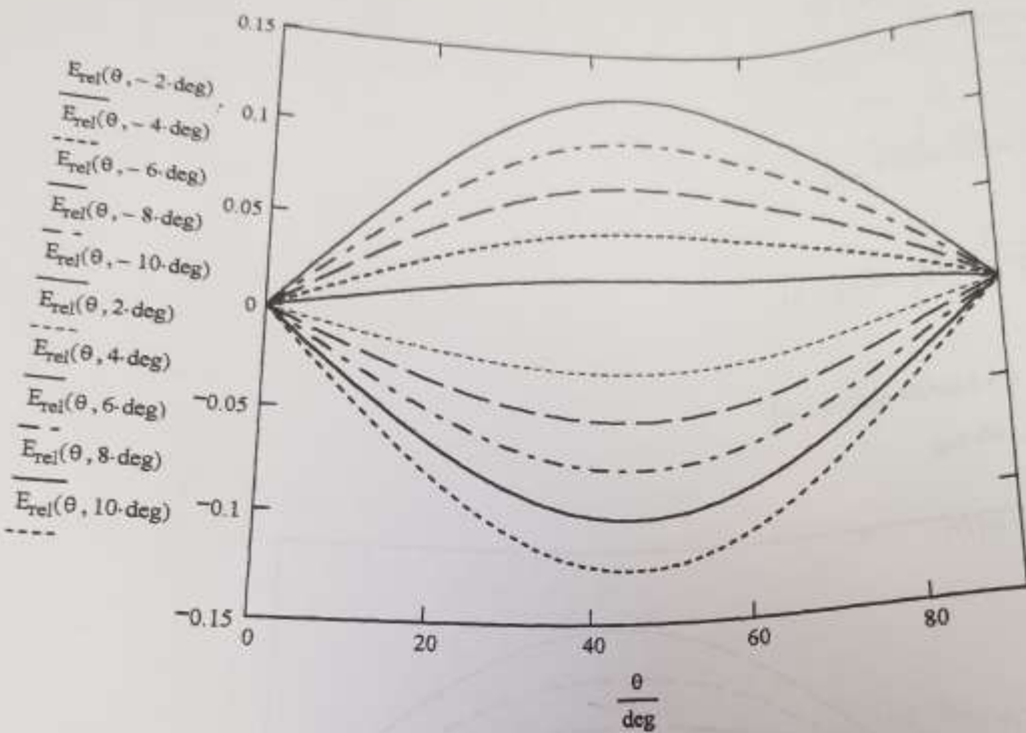
$$E_{rel}(\theta, \beta) := -(1+\nu)\cdot\sin(\beta)\cdot\sin(2\cdot\theta + \beta)$$

$$\theta := 0\cdot deg, 1\cdot deg .. 90\cdot deg$$



If Eq. (3) is used instead,

$$E_{rel}(\theta, \beta) := -(1 - \nu)\cdot\beta\cdot\sin(2\cdot\theta)$$



Problem 3-12

Input data:

$AC := 1.2 \cdot in$ $\quad CD := 1.4 \cdot in$ $\quad A'B' := 1.4038 \cdot in$ $\quad B'D' := 1.203 \cdot in$ $\quad \theta := 22.7 \cdot deg$

$\Delta\phi := .005$ $\quad \Delta\theta := .0023$

Get strain ε_{xx}:

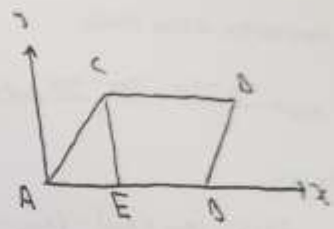
$AB := CD$ $\qquad AB = 1.4\,in$

$$\varepsilon_{xx} := \frac{A'B' - AB}{AB} \qquad \varepsilon_{xx} = 2.71429 \times 10^{-3}$$

Get strain $\varepsilon_{x'x'}$:

$A'C' := B'D'$ $\qquad A'C' = 1.203\,in$

$$\varepsilon_{x'x'} := \frac{A'C' - AC}{AC} \qquad \varepsilon_{x'x'} = 2.5 \times 10^{-3}$$

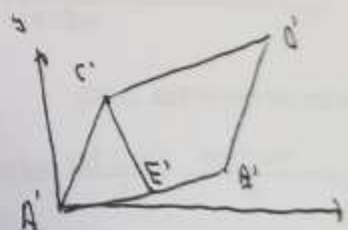


To get the strain ε_{yy}, drop a perpedicular from point C to the x axis, as shown in the figure. The vertical intersects the x axis at point E.

Get length of lines CE and AE:

$CE := AC \cdot \sin(\theta)$ $\qquad CE = 0.46309\,in$

$AE := AC \cdot \cos(\theta)$ $\qquad AE = 1.10705\,in$



In the deformed configuration, point E moves to E' and line C'E' is no longer a vertical line. However, the x and y coordinates of points C' and E' can be obtained and these can be used to get the length of line C'E'.

Get length of deformed line A'E', recalling that the strain in the element is uniform:

$A'E' := AE \cdot (1 + \varepsilon_{xx})$ $\qquad A'E' = 1.11005\,in$

Get x and y coordinates of point E':

$E'_x := A'E' \cdot \cos(\Delta\phi)$ $\qquad E'_x = 1.11004\,in$

$E'_y := A'E' \cdot \sin(\Delta\phi)$ $\qquad E'_y = 5.55023 \times 10^{-3}\,in$

Get x and y coordinates of point C':

$A'C' := B'D'$ $\qquad A'C' = 1.203\ \text{in}$

$C'_x := A'C' \cdot \cos(\theta + \Delta\theta)$ $\qquad C'_x = 1.10874\ \text{in}$

$C'_y := A'C' \cdot \sin(\theta + \Delta\theta)$ $\qquad C'_y = 0.4668\ \text{in}$

Get length of line C'E':

$$C'E' := \sqrt{(C'_x - E'_x)^2 + (C'_y - E'_y)^2} \qquad C'E' = 0.46125\ \text{in}$$

Get strain ε_{yy}:

$$\varepsilon_{yy} := \frac{C'E' - CE}{CE} \qquad \varepsilon_{yy} = -3.97194 \times 10^{-3}$$

From the first of Eqs. (3-30):

$$\varepsilon_{x'x'} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} + \frac{\varepsilon_{xx} - \varepsilon_{yy}}{2} \cdot \cos(2\cdot\theta) + \frac{1}{2} \cdot \gamma_{xy} \cdot \sin(2\cdot\theta)$$

Solve for γ_{xy}:

$$\gamma_{xy} := \frac{2\cdot\varepsilon_{x'x'} - (\varepsilon_{xx} + \varepsilon_{yy}) - (\varepsilon_{xx} - \varepsilon_{yy}) \cdot \cos(2\cdot\theta)}{\sin(2\cdot\theta)} \qquad \gamma_{xy} = 2.19501 \times 10^{-3}$$

Use the last two of Eqs. (3-30):

$$\varepsilon_{y'y'} := \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} - \frac{\varepsilon_{xx} - \varepsilon_{yy}}{2} \cdot \cos(2\cdot\theta) - \frac{1}{2} \cdot \gamma_{xy} \cdot \sin(2\cdot\theta) \qquad \varepsilon_{y'y'} = -3.75766 \times 10^{-3}$$

$$\gamma_{x'y'} := -(\varepsilon_{xx} - \varepsilon_{yy}) \cdot \sin(2\cdot\theta) + \gamma_{xy} \cdot \cos(2\cdot\theta) \qquad \gamma_{x'y'} = -3.21954 \times 10^{-3}$$

Is Eq. (3-52) satisfied?

$$\varepsilon_{xx} + \varepsilon_{yy} - (\varepsilon_{x'x'} + \varepsilon_{y'y'}) = 0$$

Problem 3-13

From Problem 3-12:

$\varepsilon_{xx} := .00271$ $\qquad\varepsilon_{yy} := -.00397$ $\qquad\gamma_{xy} := .00220$

The principal strains are given by Eqs. (3-42):

$$\varepsilon_1 := \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} + \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{1}{2}\cdot\gamma_{xy}\right)^2} \qquad \varepsilon_1 = 2.886 \times 10^{-3}$$

$$\varepsilon_2 := \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} - \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{1}{2}\cdot\gamma_{xy}\right)^2} \qquad \varepsilon_2 = -4.146 \times 10^{-3}$$

The principal directions are given by Eq. (3-43):

$$\tan(\theta) = \frac{\varepsilon - \varepsilon_{xx}}{\frac{1}{2}\cdot\gamma_{xy}}$$

Therefore,

$$\theta_1 := \operatorname{atan2}\left(\frac{1}{2}\cdot\gamma_{xy}, \varepsilon_1 - \varepsilon_{xx}\right) \qquad \theta_1 = 9.114\,\text{deg}$$

$$\theta_2 := \operatorname{atan2}\left(\frac{1}{2}\cdot\gamma_{xy}, \varepsilon_2 - \varepsilon_{xx}\right) \qquad \theta_2 = -80.886\,\text{deg}$$

#3-14

$$u(x,y) = a_0 + a_1 x + a_2 y + a_3 xy$$

$$v(x,y) = a_4 + a_5 x + a_6 y + a_7 xy$$

Therefore, with Eqs (3-6) - (3-9),

$$\epsilon_x = \frac{\partial u}{\partial x} = a_1 + a_3 y$$

$$\epsilon_y = \frac{\partial v}{\partial y} = a_6 + a_7 x$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = a_2 + a_3 x + a_5 + a_7 y$$

In matrix form these become,

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 1 & x & 0 & 1 & 0 & y \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{Bmatrix}$$

Note that $x_i = y_i = y_j = x_l = 0$ and $x_j = x_k$, $y_k = y_l$

$$u_i = a_0$$

$$v_i = a_4$$

$$u_j = a_0 + a_1 x_j \qquad \text{eqn (5)}$$

$$v_j = a_4 + a_5 x_j$$

$$\vdots$$

or, in matrix form,

$$
\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \\ u_l \\ v_l \end{Bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & x_j & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & x_j & 0 & 0 \\
1 & x_j & y_k & x_j y_k & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & x_j & y_k & x_j y_k \\
1 & 0 & y_k & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & y_k & 0
\end{bmatrix}
\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{Bmatrix}
$$

These can be rearranged to give

$$
\begin{Bmatrix} u_i \\ u_j \\ u_k \\ u_l \\ v_i \\ v_j \\ v_k \\ v_l \end{Bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & x_j & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & x_j & y_k & x_j y_k & 0 & 0 & 0 & 0 \\
1 & 0 & y_k & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & x_j & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & x_j & y_k & x_j y_k \\
0 & 0 & 0 & 0 & 1 & 0 & y_k & 0
\end{bmatrix}
\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{Bmatrix}
$$

These equations are of the form,

$$\begin{Bmatrix}\{u\}_{1node}\\ \{u\}_{2node}\end{Bmatrix} = \begin{bmatrix}[x]_{node} & [0]\\ [0] & [x]_{node}\end{bmatrix}\begin{Bmatrix}\{a\}_{1node}\\ \{a\}_{2node}\end{Bmatrix}$$

where
$$\{u\}_{1node} = [u_i \;\; u_j \;\; u_k \;\; u_l]^T$$
$$\{u\}_{2node} = [v_i \;\; v_j \;\; v_k \;\; v_l]^T$$
$$[x]_{node} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & x_j & 0 & 0 \\ 1 & x_j & y_k & x_j y_k \\ 1 & 0 & y_k & 0 \end{bmatrix}$$
$$\{a\}_{1node} = [a_0 \;\; a_1 \;\; a_2 \;\; a_3]^T$$
$$\{a\}_{2node} = [a_4 \;\; a_5 \;\; a_6 \;\; a_7]^T$$

$$\{a\}_{1node} = [x]^{-1}_{node}\{u\}_{1node}$$
$$\{a\}_{2node} = [x]^{-1}_{node}\{u\}_{2node}$$

or $$\begin{Bmatrix} \{a\}_{1,node} \\ \{a\}_{2,node} \end{Bmatrix} = \begin{bmatrix} [x]^{-1}_{node} & [0] \\ [0] & [x]^{-1}_{node} \end{bmatrix} \begin{Bmatrix} \{u\}_{1,node} \\ \{u\}_{2,node} \end{Bmatrix}$$

$[x]^{-1}_{node}$ is computed as

$$[x]^{-1}_{node} = \frac{1}{x_j y_k} \begin{bmatrix} x_j y_k & 0 & 0 & 0 \\ -y_k & y_k & 0 & 0 \\ -x_j & 0 & 0 & x_j \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

Note that $x_j y_k$ is the area of the rectangular element.
With this,

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{Bmatrix} = \frac{1}{x_j y_k} \begin{bmatrix} x_j y_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -y_k & y_k & 0 & 0 & 0 & 0 & 0 & 0 \\ -x_j & 0 & 0 & x_j & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_j y_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -y_k & y_k & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_j & 0 & 0 & x_j \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \\ u_k \\ u_l \\ v_i \\ v_j \\ v_k \\ v_l \end{Bmatrix}$$

[illegible]

The nodal displacement vector is now rearranged to [illegible] to the original order, u_i, v_i, u_j, v_j, etc.

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{Bmatrix} = \frac{1}{x_j y_k} \begin{bmatrix} x_j y_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -y_k & 0 & y_k & 0 & 0 & 0 & 0 & 0 \\ -x_j & 0 & 0 & 0 & 0 & 0 & x_j & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & x_j y_k & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -y_k & 0 & y_k & 0 & 0 & 0 & 0 \\ 0 & -x_j & 0 & 0 & 0 & 0 & 0 & x_j \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \\ u_l \\ v_l \end{Bmatrix}$$

This is of the form,

$$\{a\} = [C]^{-1} \{U_{node}\}$$

where $\{a\} = [a_0 \ a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ a_6 \ a_7]^T$

$$\{U_{node}\} = [u_i \ v_i \ u_j \ v_j \ u_k \ v_k \ u_l \ v_l]^T$$

$$[C] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & x_j & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_j & 0 & 0 \\ 1 & x_j & y_k & x_j y_k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_j & y_k & x_j y_k \\ 1 & 0 & y_k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & y_k & 0 \end{bmatrix}$$

$$[C]^{-1} = \frac{1}{x_j y_k} \begin{bmatrix} x_j y_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -y_k & 0 & y_k & 0 & 0 & 0 & 0 & 0 \\ -x_j & 0 & 0 & 0 & 0 & 0 & x_j & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & x_j y_k & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -y_k & 0 & y_k & 0 & 0 & 0 & 0 \\ 0 & -x_j & 0 & 0 & 0 & 0 & 0 & x_j \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \end{bmatrix}$$

$$\{\epsilon\} = [x][C]^{-1}\{u_{node}\}$$

where

$$[x] = \begin{bmatrix} 0 & 1 & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 1 & x & 0 & 1 & 0 & y \end{bmatrix}$$