Homework 2 Solutions

Section 2-1

2-1. Derive Eqs.(2-5).

We follow the procedure used to derive Eq.(2-4) (see Fig.2-2 where the values of stress components are properly modified)).

FACE	MOMENT ABOUT y-AXIS
BCGF	$\underbrace{(\sigma_{xx} + d\sigma_{xx})}_{stress}\underbrace{dy\,dz}_{area}\underbrace{\frac{dz}{2}}_{moment\ arm} - (\sigma_{xz} + d\sigma_{xz})dy\,dz\,dx$
CDHG	$-(\sigma_{yz} + d\sigma_{yz})\frac{dx}{2}dx\,dz + (\sigma_{yx} + d\sigma_{yx})\frac{dz}{2}dx\,dz$
ADHE	$-\sigma_{xx}\frac{dz}{2}dy\,dz$
AEFB	$\sigma_{yz}\frac{dx}{2}dx\,dz - \sigma_{yx}\frac{dz}{2}dx\,dz$
ABCD	$\sigma_{zz}\frac{dx}{2}dx\,dy$
FGHE	$-(\sigma_{zz} + d\sigma_{zz})\frac{dx}{2}dx\,dy + (\sigma_{zx} + d\sigma_{zx})dz\,dx\,dy$

Summing up these expressions and adding moments due to the body forces we obtain:

$$(\sigma_{xx} + d\sigma_{xx})dz\,dy\frac{dz}{2} - (\sigma_{xz} + d\sigma_{xz})dx\,dy\,dz - (\sigma_{yz} + d\sigma_{yz})\frac{dx}{2}dx\,dz$$

$$+(\sigma_{yx} + d\sigma_{yx})\frac{dz}{2}dx\,dz - \sigma_{xx}\frac{dz}{2}dx\,dz + \sigma_{yz}\frac{dx}{2}dx\,dz - \sigma_{yx}\frac{dz}{2}dx\,dz$$

$$+\sigma_{zz}\frac{dx}{2}dx\,dy - (\sigma_{zz} + d\sigma_{zz})\frac{dx}{2}dx\,dy + (\sigma_{zx} + d\sigma_{zx})dz\,dx\,dy$$

$$+b_x dx\,dy\,dz\frac{dz}{2} - b_z dx\,dy\,dz\frac{dx}{2}.$$

Next we substitute these expressions into Eq.(2-3):

$$\sum M_y = 0,$$

On discarding smaller order terms and dividing this equation through by $dx\,dy\,dz$ we get

$$\sigma_{xz} = \sigma_{zx} \qquad (2\text{-}5)$$

Similarly

FACE	MOMENT ABOUT z-AXIS
BCGF	$-\underbrace{(\sigma_{xx} + d\sigma_{xx})}_{stress}\underbrace{dy\,dz}_{area}\underbrace{\frac{dy}{2}}_{moment\ arm} + (\sigma_{xy} + d\sigma_{xy})\,dy\,dz\,dx$
CDHG	$(\sigma_{yy} + d\sigma_{yy})\frac{dx}{2}dx\,dz - (\sigma_{yx} + d\sigma_{yx})dy\,dx\,dz$
ADHE	$\sigma_{xx}\frac{dy}{2}dy\,dz$
AEFB	$-\sigma_{yy}\frac{dx}{2}dx\,dz$
ABCD	$\sigma_{zx}\frac{dy}{2}dx\,dy - \sigma_{zy}\frac{dx}{2}dx\,dy$
FGHE	$-(\sigma_{zx} + d\sigma_{zx})\frac{dy}{2}dx\,dy + (\sigma_{zy} + d\sigma_{yx})\frac{dx}{2}dx\,dy$

Summing up these expressions and adding moments due to the body forces we obtain:

$$-(\sigma_{xx}+d\sigma_{xx})dz\,dy\frac{dy}{2}+(\sigma_{xy}+d\sigma_{xy})dx\,dy\,dz$$
$$+(\sigma_{yy}+d\sigma_{yy})\frac{dx}{2}dx\,dz-(\sigma_{yx}+d\sigma_{yx})dy\,dx\,dz$$
$$+\sigma_{xx}\frac{dy}{2}dy\,dz-\sigma_{yy}\frac{dx}{2}dx\,dz+\sigma_{zx}\frac{dy}{2}dx\,dy-\sigma_{zy}\frac{dx}{2}dx\,dy$$
$$-(\sigma_{zx}+d\sigma_{zx})\frac{dy}{2}dx\,dy+(\sigma_{zy}+d\sigma_{zy})\frac{dx}{2}dx\,dy$$
$$-b_x dx\,dy\,dz\frac{dy}{2}+b_y dx\,dy\,dz\frac{dx}{2}.$$

Next we substitute these expressions into Eq.(2-3):

$$\sum M_z=0.$$

On discarding smaller order terms and dividing this equation through by $dx\,dy\,dz$ we get

$$\sigma_{xy}=\sigma_{yx} \qquad (2\text{-}5)$$

2-2. Show all steps in the derivation of Eq.(2-13).
Substituting Eq.(2-8) into Eq.(2-12) we get

$$\boldsymbol{t}^{(n)}\Delta A+\boldsymbol{t}^{(-x)}n_x\Delta A+\boldsymbol{t}^{(-y)}n_y\Delta A+\boldsymbol{t}^{(-z)}n_z\Delta A+\frac{h}{2}\Delta A\boldsymbol{b}=0$$

In the limit, with $h\to 0$ and taking into account that $\boldsymbol{t}^{(-x)}=-\boldsymbol{t}^{(x)}$ etc, we find that:

$$\boldsymbol{t}^{(n)}=\boldsymbol{t}^{(x)}n_x+\boldsymbol{t}^{(y)}n_y+\boldsymbol{t}^{(z)}n_z \qquad (2\text{-}13)$$

2-3. Derive Eq.(2-17), then write it as a product of three matrices.
Substituting Eq.(2-7) into Eq.(2-16) we get

$$\sigma_n=[\boldsymbol{t}^{(x)}\cdot(n_x\boldsymbol{i}+n_y\boldsymbol{j}+n_z\boldsymbol{k})]n_x+[\boldsymbol{t}^{(y)}\cdot(n_x\boldsymbol{i}+n_y\boldsymbol{j}+n_z\boldsymbol{k})\boldsymbol{t}^{(x)}]n_y+$$
$$+[\boldsymbol{t}^{(z)}\cdot(n_x\boldsymbol{i}+n_y\boldsymbol{j}+n_z\boldsymbol{k}))]n_z$$

Using Eq.(2-1) to replace the dot products leads to

... start with the equation (2-30) which when multiplied by i gives

$$\boldsymbol{s} \cdot \boldsymbol{i} = (\boldsymbol{t}^{(x)} \cdot \boldsymbol{i}\, n_x + \boldsymbol{t}^{(y)} \cdot \boldsymbol{i}\, n_y + \boldsymbol{t}^{(z)} \cdot \boldsymbol{i}\, n_z - \sigma_n \boldsymbol{n} \cdot \mathbf{i}) / \tau$$

Using here Eq.(2-1) and Eq.(2-7) we get the first of Eqs.(2-31)

$$s_x = (\sigma_{xx} n_x + \sigma_{xy} n_y + \sigma_{xz} n_z - \sigma_n n_x)/\tau,$$

The other two are obtained in the same way.

2-7. The stress matrix S at a point M is given by

$$S = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

Draw a plane T through M with the direction cosines:

$$n_x = \frac{1}{\sqrt{3}}, \quad n_y = \frac{1}{\sqrt{3}}, \quad n_z = \frac{1}{\sqrt{3}}.$$

Find:
(a) the traction vector $t^{(n)}$ acting on T at M,
(b) its Cartesian components $t_x^{(n)}$, $t_y^{(n)}$, and $t_z^{(n)}$
(c) its normal component σ_n

We follow the procedure outlined in Example 2-4.

(a) First we identify the stress components from the matrix S: $s_{xx}=2$, $s_{xy}=1$, $s_{xz}=2$, $s_{yy}=3$, $s_{yz}=0$, $s_{zz}=2$. Next we evaluate the traction vector acting at M:

$$t^{(n)} = t_x^{(n)}\,\boldsymbol{i} + t_y^{(n)}\,\boldsymbol{j} + t_z^{(n)}\,\boldsymbol{k}$$

where the components of $t^{(n)}$ are given by (Eq.2-14). Thus

$$t^{(n)} = (\sigma_{xx}n_x + \sigma_{xy}n_y + \sigma_{xz}n_z)\,\boldsymbol{i} + (\sigma_{xy}n_x + \sigma_{yy}n_y + \sigma_{yz}n_z)\,\boldsymbol{j}$$
$$+(\sigma_{xz}n_x + \sigma_{yz}n_y + \sigma_{zz}n_z)\,\boldsymbol{j}$$

and finally, substituting the numerical values for the stress components and direction cosines we get

$$t^{(n)} = (5\boldsymbol{i} + 4\,\boldsymbol{j} + 4\boldsymbol{k})\;/\;\sqrt{3}$$

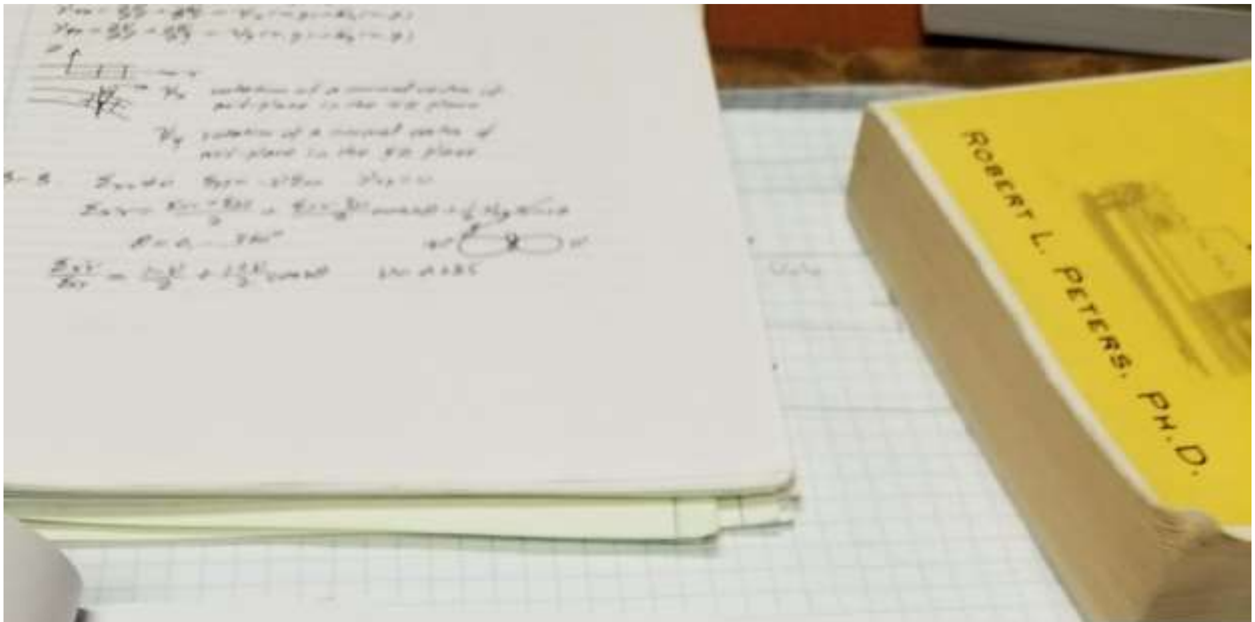
(b) Now :

$$t_x^{(n)} = (2+1+2)/\sqrt{3} = 5/\sqrt{3}, t_y^{(n)} = (1+3+0)/\sqrt{3} = 4/\sqrt{3},$$
$$t_z^{(n)} = (2+0+2)/\sqrt{3} = 4/\sqrt{3}$$

(c) Substituting the data into Eq.(2-17) we get

$$\sigma_n = (2+3+2+2\cdot 1+2\cdot 2+2\cdot 0)\cdot\frac{1}{3} = 13/3$$

2-8. A solid, rectangular bar is subject to uniform tension q per unit area (Fig.P2-8). The state of stress inside the bar was found to be also uniform such that $\sigma_{xx} = q$, and all other stress components equal to zero. Find the normal component σ_n, and the tangential component τ, of the traction $t^{(n)}$ acting at an inside point on the plane with the outer normal



$$N = i + 2j + k.$$

<u>Hint</u>:

$$n = \frac{N}{|N|}$$

We have

$$n = (i + 2j + k) / \sqrt{1 + 4 + 1} = (i + 2j + k) / \sqrt{6}$$

Hence: $n_x = 1/\sqrt{6}$, $n_y = 2/\sqrt{6}$, $n_z = 1/\sqrt{6}$

Using Eq.(2-17) where we substitute the known values of stress components we get: $s_n = q/6$.

Finally, from Eq.(2-27) we obtain

$$\tau = \sqrt{q^2 / 6 - q^2 / 36} = \frac{\sqrt{5}}{6} q$$