MECH 5312 – Solid Mechanics II

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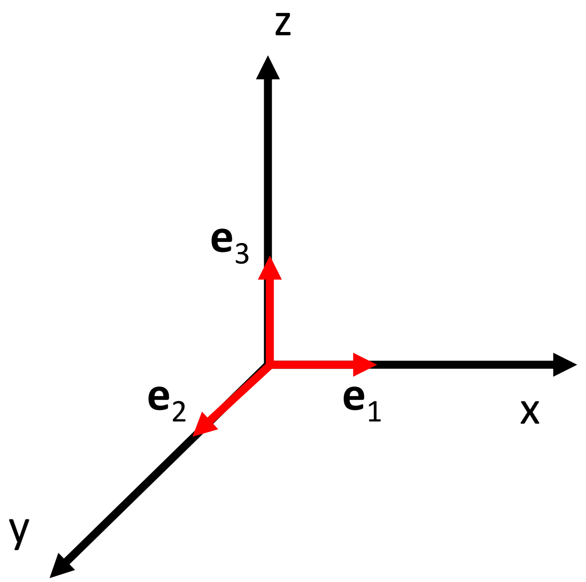
Preliminary Math

- **Base Vectors** – Describe the coordinates system in 3D space.
- Base Vectors are Orthogonal to each other. (i.e. the normal of the plane formed by two base vectors is always the third base vector.)
- Let's assume the unit direction vector $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are directed along the positive x, y, and z axes.

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



- Note: base vectors are not always aligned to x, y, z.

Preliminary Math

- **Scalar/Dot Product** – the produce of the magnitude of one vector and the component of the second vector in the direction of the first.

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$$

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$$

$$\mathbf{u}\cdot\mathbf{v} = \left(u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3\right)\cdot\left(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3\right)$$

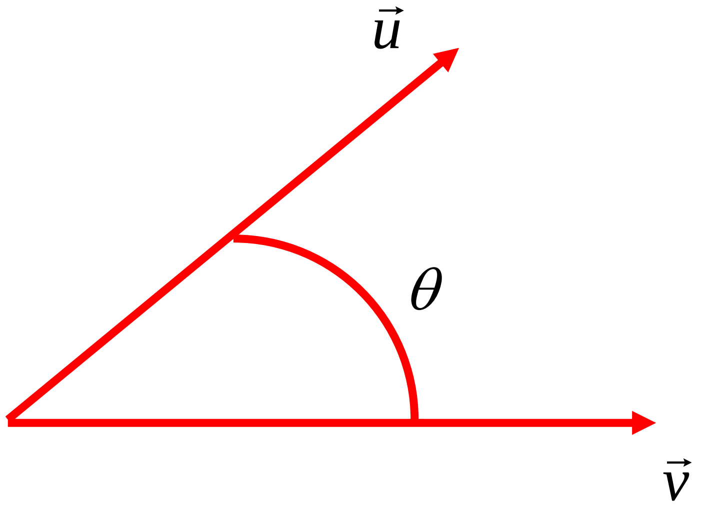
$$= \left(u_i\mathbf{e}_i\right)\cdot\left(v_j\mathbf{e}_j\right)$$

$$= u_i v_j\left(\mathbf{e}_i\right)\cdot\left(\mathbf{e}_j\right)$$

$$= u_i v_i$$

$$= \left(u_1v_1 + u_2v_2 + u_3v_3\right)$$

$$\boxed{\vec{u}\cdot\vec{v} = \left(u_1v_1 + u_2v_2 + u_3v_3\right)}$$ yields a scalar quantity.



$$\boxed{\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta}$$

where θ is the angle between the vectors

Preliminary Math

- Often, $\mathbf{e}_1 = \hat{i} \quad \mathbf{e}_2 = \hat{j} \quad \mathbf{e}_3 = \hat{k}$
- Thus, $\boxed{\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}}$ where $\delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\boxed{\mathbf{I} = \delta_{ij}}$ Identity Matrix / Kronecker Delta

Preliminary Math

- **Tensor Multiplication**
- Tensor, **A**

Row x Column

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

3x3

- Vector, **b**

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

3x1

- Product of a Tensor and a Vector is a Vector!

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \left(a_{21}b_1 + a_{22}b_2 + a_{23}b_3\right) \\ \left(a_{21}b_1 + a_{22}b_2 + a_{23}b_3\right) \\ \left(a_{21}b_1 + a_{22}b_2 + a_{23}b_3\right) \end{bmatrix}$$

3x3 3x1 3x1

Preliminary Math

- **Tensor Multiplication (continued)**

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} =$$

$$\begin{bmatrix} \left(a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}\right) & \left(a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}\right) & \left(a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33}\right) \\ \left(a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}\right) & \left(a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}\right) & \left(a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}\right) \\ \left(a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31}\right) & \left(a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32}\right) & \left(a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33}\right) \end{bmatrix}$$

Example

Example 1

- Given the following orthogonal base vectors.

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ \dfrac{\sqrt{2}}{2} \\ \dfrac{\sqrt{2}}{2} \end{bmatrix} \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ -\dfrac{\sqrt{2}}{2} \\ \dfrac{\sqrt{2}}{2} \end{bmatrix}$$

- Prove $\boxed{\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}}$

Example 2

- Show that the dot product of $\mathbf{u} \cdot \mathbf{i}, \quad \mathbf{u} \cdot \mathbf{j}, \quad \mathbf{u} \cdot \mathbf{k}$

- are equal to the projected component of each vector in the index directions.

Example 3

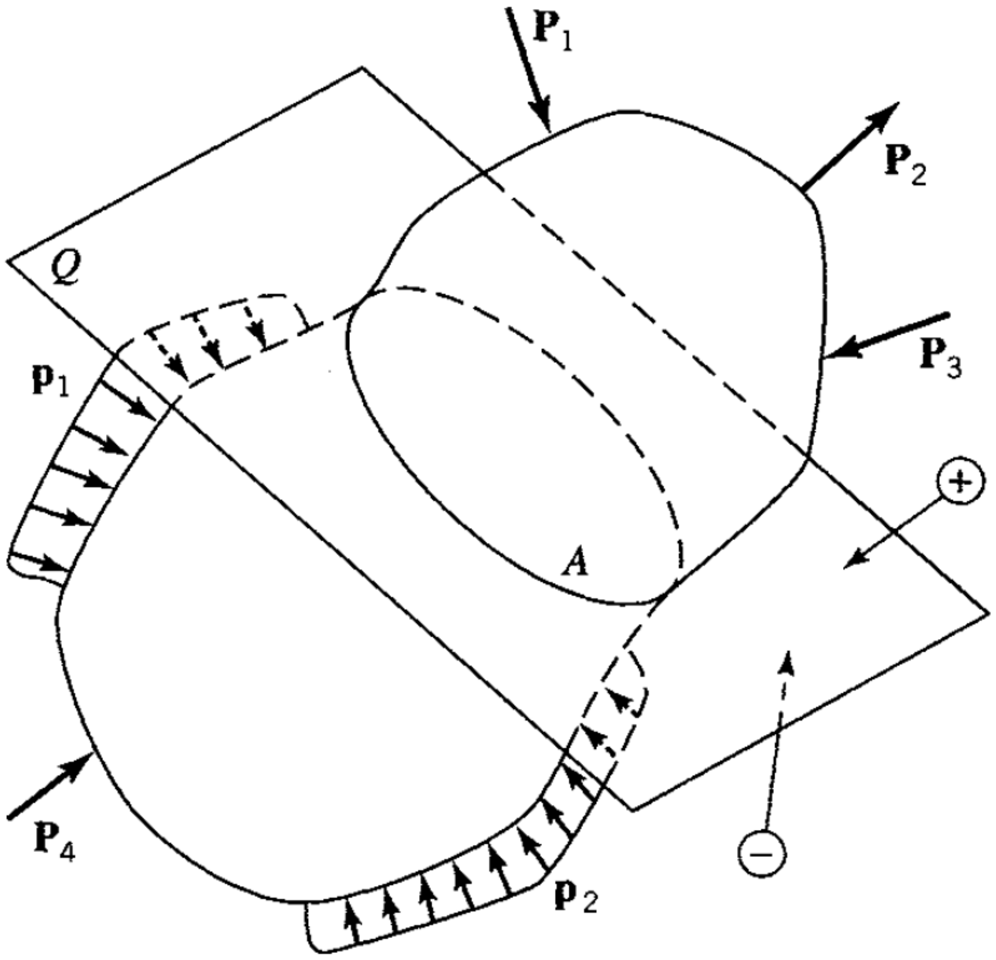
- Given

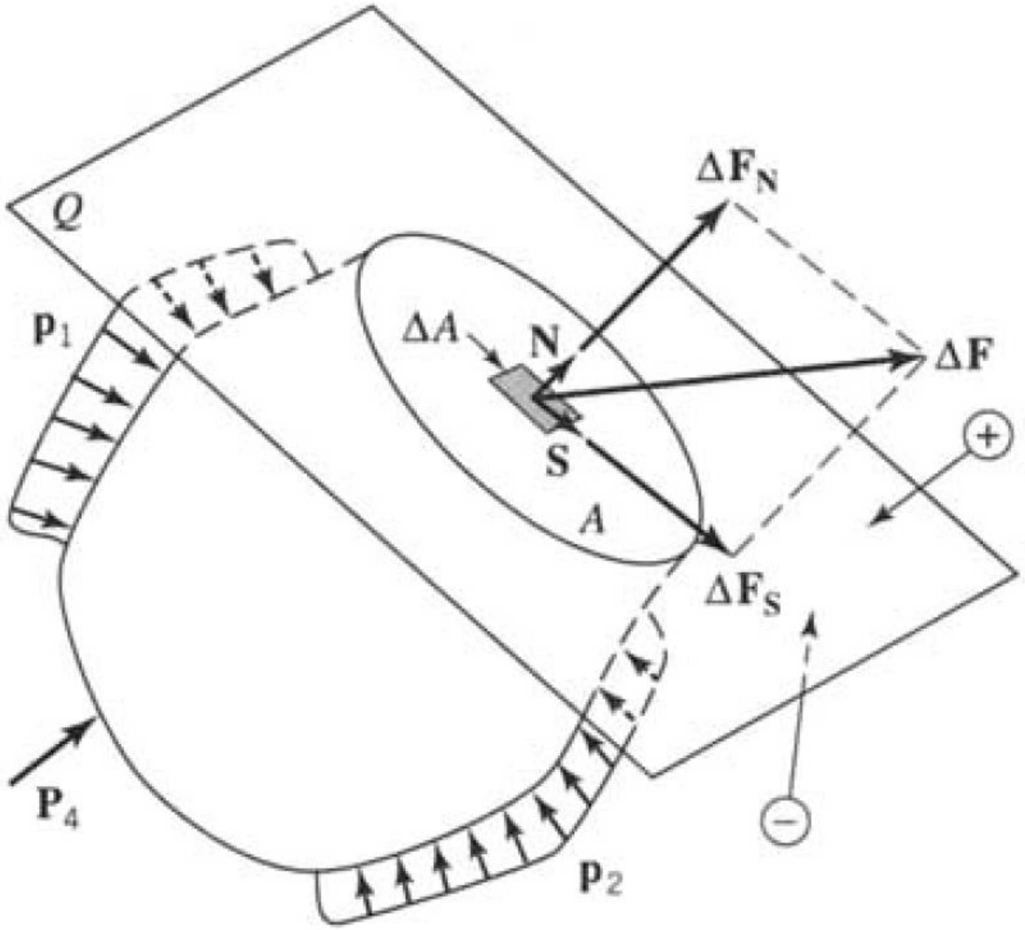
$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

- Find,

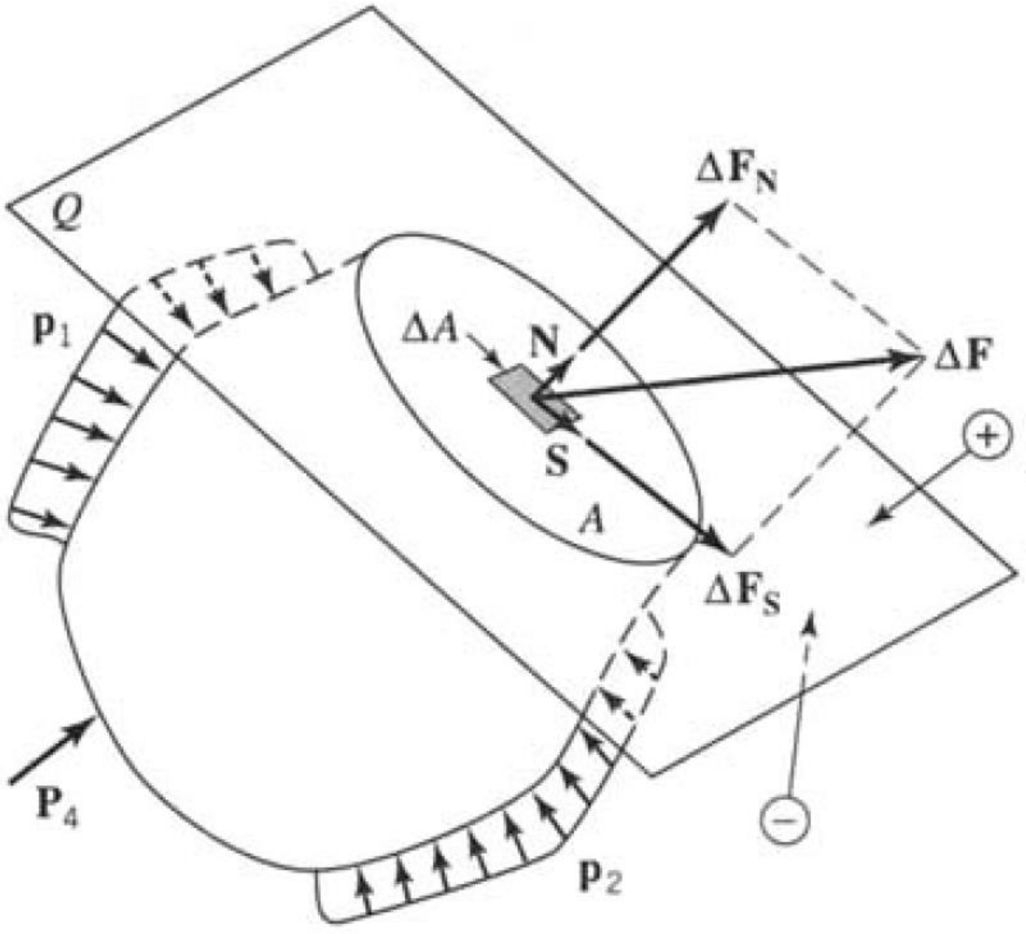
$$\mathbf{A} \cdot \mathbf{b}$$

Concept of Stress





Concept of Stress



- Traction Vector, **t**

$$\boxed{\mathbf{t} = \lim_{\Delta A \to \infty} \frac{\Delta \mathbf{F}}{\Delta A}}$$

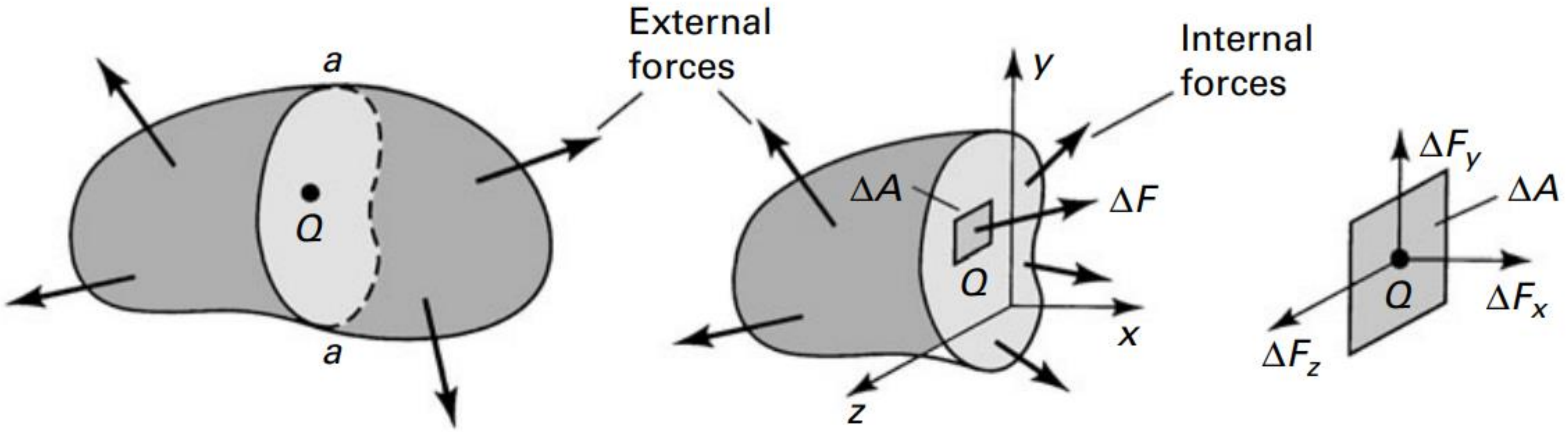
Traction can be decomposed into Normal and Shear Components

$$\boxed{\mathbf{t} = \sigma_n \mathbf{n} + \tau \mathbf{s}}$$

$$\boxed{\sigma_n = \lim_{\Delta A \to \infty} \frac{\Delta \mathbf{F}_n}{\Delta A}} \qquad \boxed{\tau = \lim_{\Delta A \to \infty} \frac{\Delta \mathbf{F}_s}{\Delta A}}$$

Concept of Stress

- Traction relates to Cauchy Stresses when the normal of a surface aligns with a base vector. For example, the normal aligned with x axis.

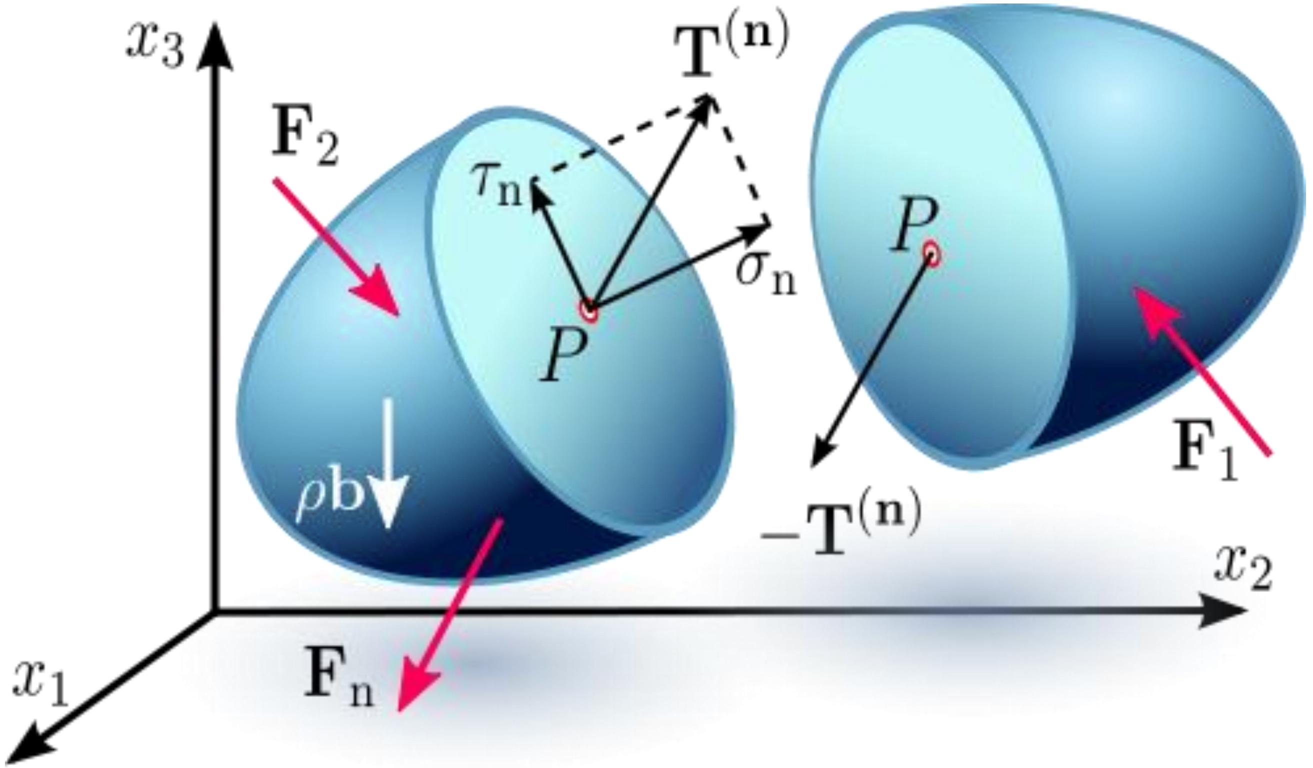


$$\sigma_{xx} = \lim_{\Delta A \to \infty} \frac{\Delta \mathbf{F}_x}{\Delta A}, \qquad \sigma_{xy} = \lim_{\Delta A \to \infty} \frac{\Delta \mathbf{F}_y}{\Delta A}, \qquad \sigma_{xz} = \lim_{\Delta A \to \infty} \frac{\Delta \mathbf{F}_z}{\Delta A}$$

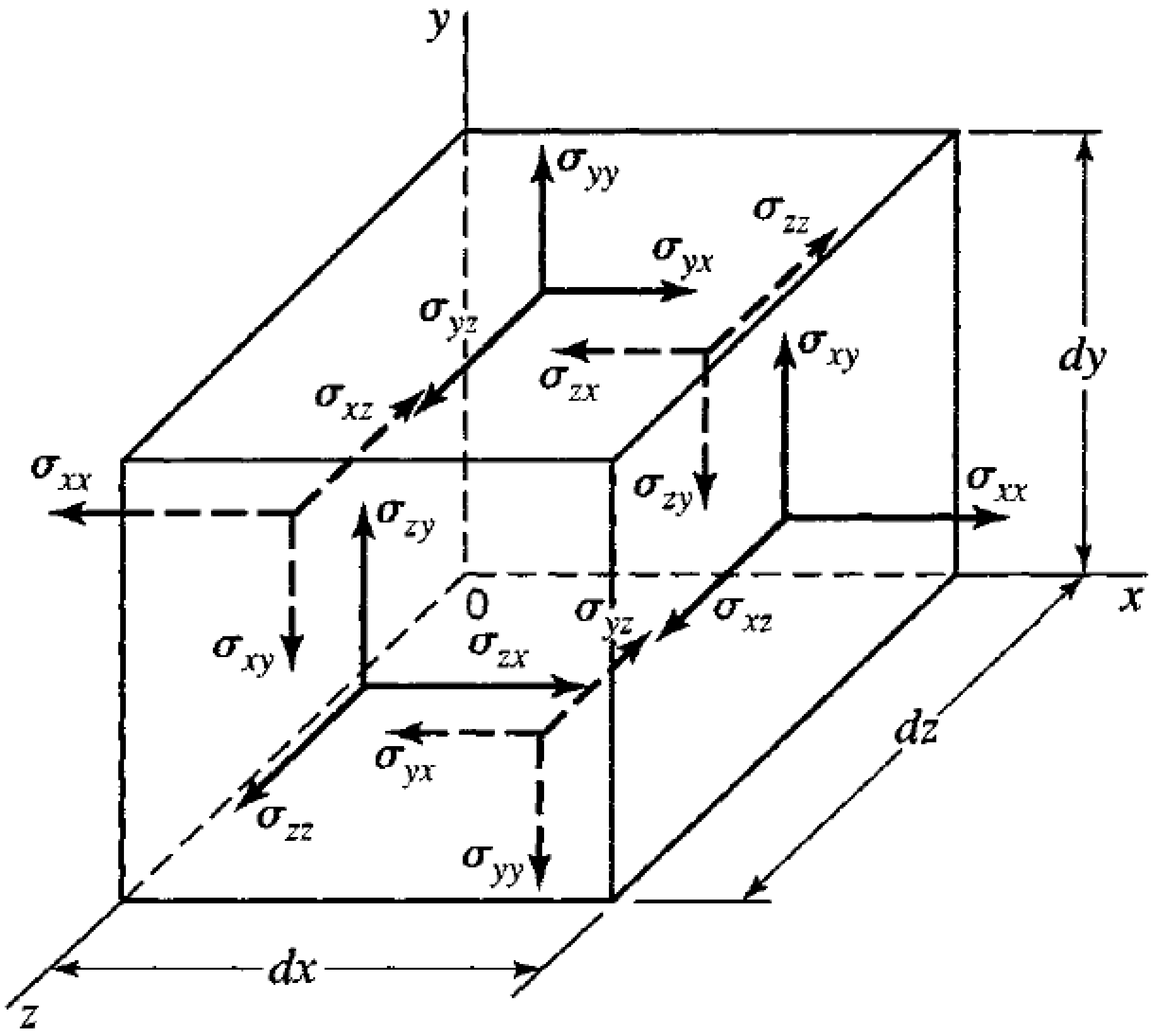
Example

Example 4

- Using static equilibrium to prove that the sum of internal forces across the two halves are equal and opposite. (i.e. internal forces disappear when we remove the section!) *Cauchy's Fundamental Lemma*



Stress Components



Cauchy Stress Tensor, **S**

$$\mathbf{S} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

or

$$\mathbf{S} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Solecki Boresi

Notation -> $\mathbf{S} = \boldsymbol{\sigma} = \mathbf{T}$

Stewart

Stress Components

- If we know the tractions , $\mathbf{t}^{(x)}$, $\mathbf{t}^{(y)}$, and $\mathbf{t}^{(z)}$ on the x, y, and z surfaces of a 3D element.

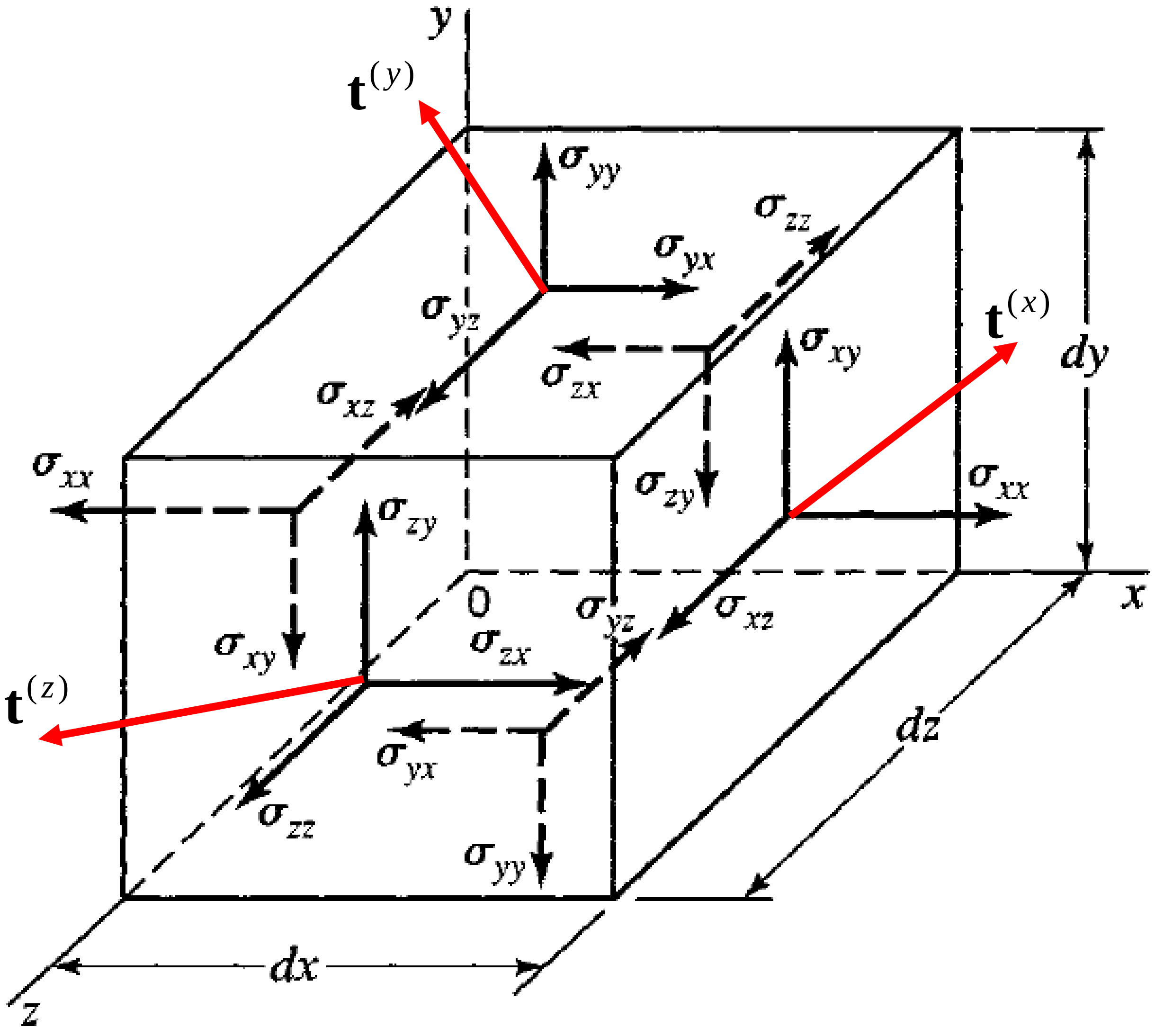
- We can find the Cauchy stress components as follows

$$\sigma_{xx} = \mathbf{t}^{(x)} \cdot i, \qquad \sigma_{xy} = \mathbf{t}^{(x)} \cdot j, \qquad \sigma_{xz} = \mathbf{t}^{(x)} \cdot k$$

$$\sigma_{yx} = \mathbf{t}^{(y)} \cdot i, \qquad \sigma_{yy} = \mathbf{t}^{(y)} \cdot j, \qquad \sigma_{yz} = \mathbf{t}^{(y)} \cdot k$$

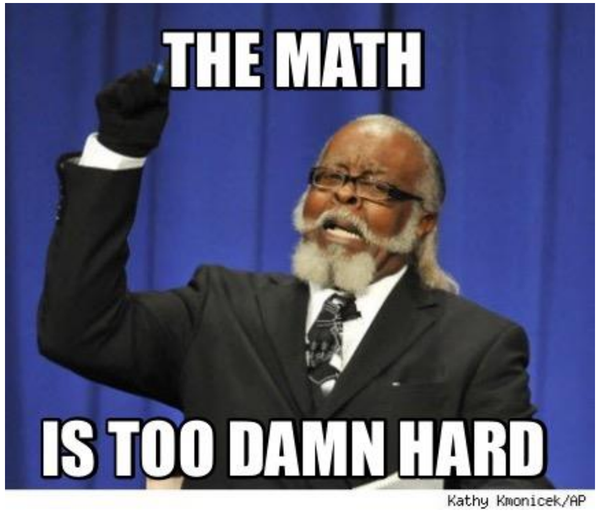
$$\sigma_{zx} = \mathbf{t}^{(z)} \cdot i, \qquad \sigma_{zy} = \mathbf{t}^{(z)} \cdot j, \qquad \sigma_{zz} = \mathbf{t}^{(z)} \cdot k$$

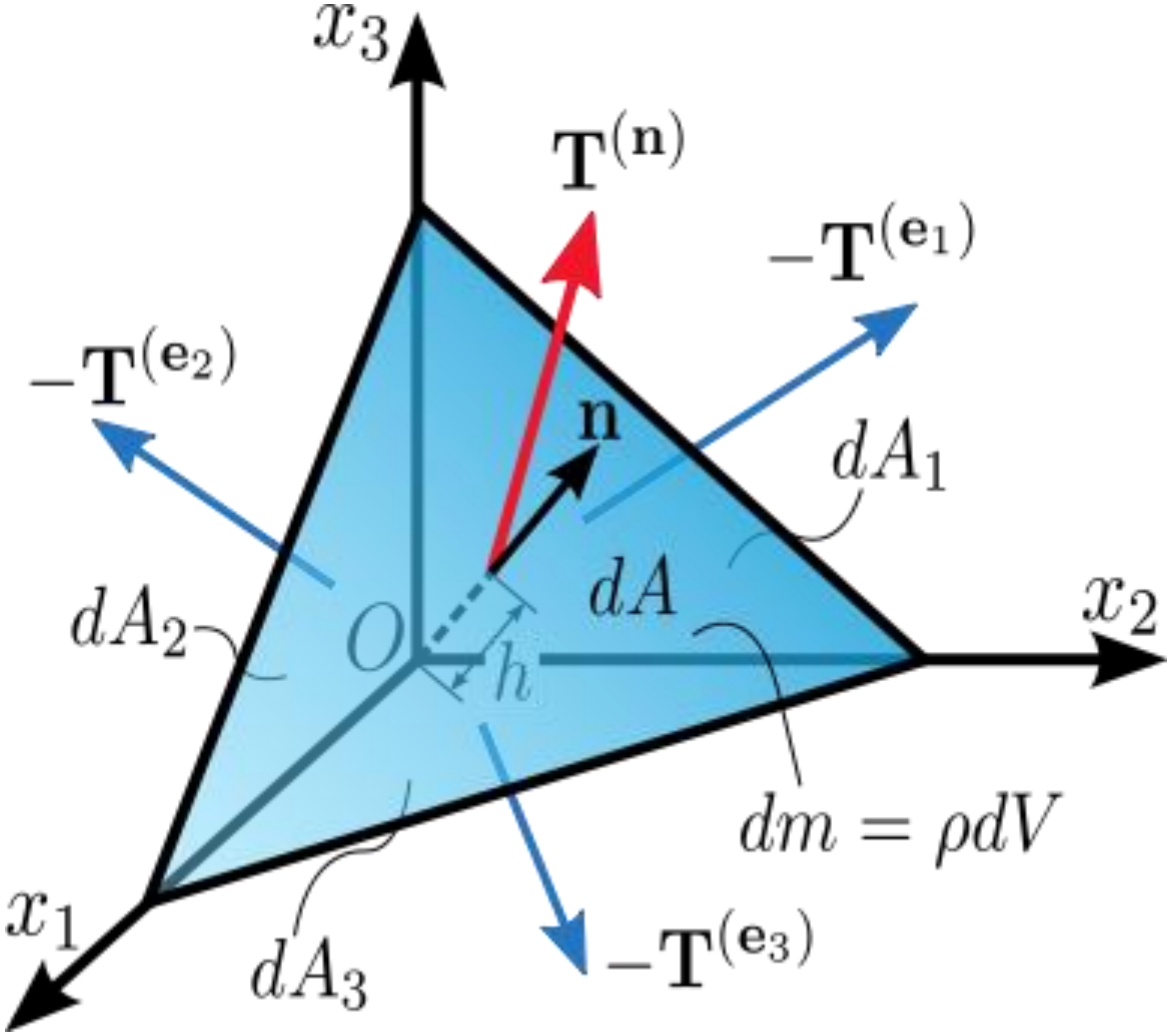
Surface Direction



Stress Components

- We can also determine the relationship between traction, **t**$^{(n)}$ on an arbitrary plane, **n** and Cauchy stress by invoking Equilibrium!





Derive!

Stress Components

- Thus the relationship between the traction on an arbitrary plane, **n** and Cauchy stress equates to

$$\boxed{\mathbf{t}^{(n)} = \mathbf{S} \cdot \mathbf{n}}$$

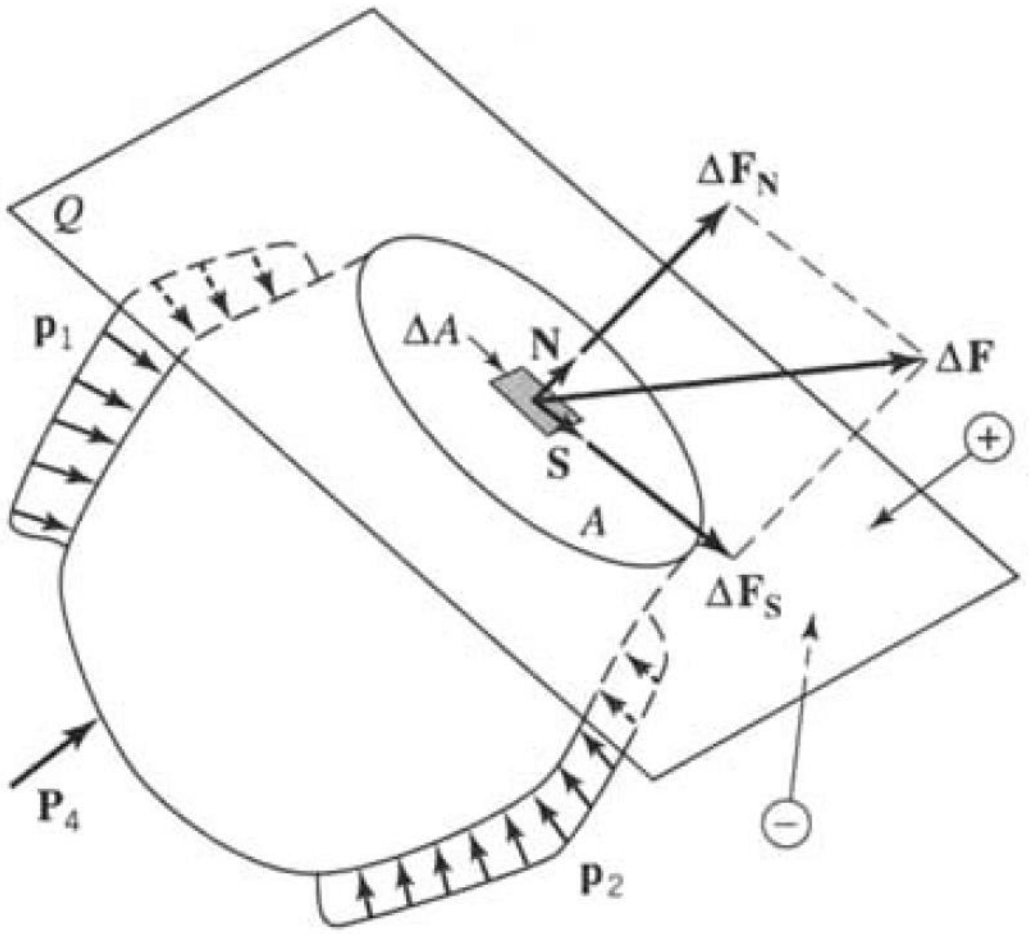
Concept of Stress

Traction can be decomposed into Normal and Shear Components

$$\boxed{\mathbf{t}^{(n)} = \sigma_n \mathbf{n} + \tau \mathbf{s}}$$

$$\boxed{\sigma_n = \mathbf{t} \cdot \mathbf{n}}$$

$$\boxed{\tau = \left(\mathbf{t}^{(n)} \cdot \mathbf{t}^{(n)} - \sigma_n^2\right)^{1/2}}$$



Example 5

- Select a σ_{yy} such that there will be a traction free plane

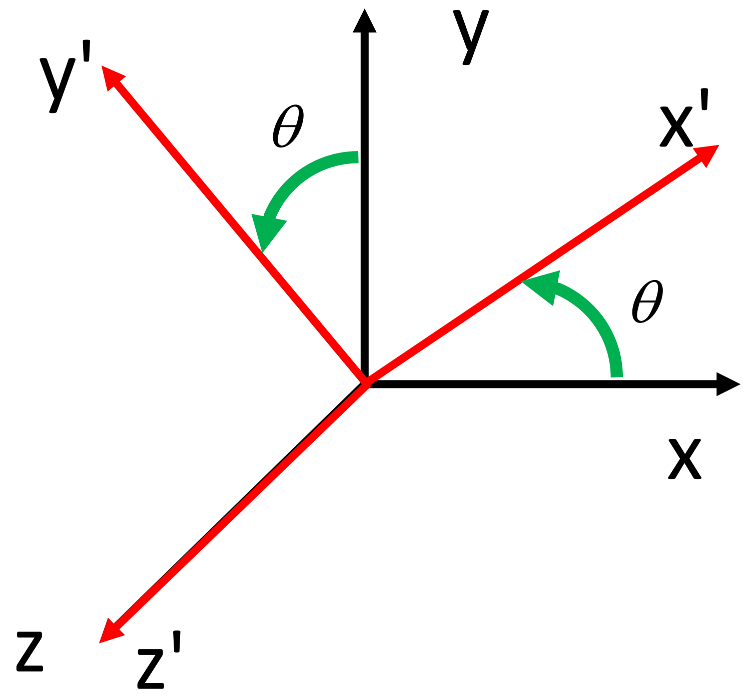
$$\boldsymbol{\sigma} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sigma_{yy} & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

Example 6

2-4 from Solecki

Transformation of Stress Components

- When the reference coordinate system is rotated.
- The state of stress will transform.
- For example, rotating about z
- x, y, z to x’, y’, z’



$$n_{x'x} = \cos(\theta), \quad n_{x'y} = \sin(\theta), \quad n_{x'z} = 0$$

$$n_{y'x} = -\sin(\theta), \quad n_{y'y} = \cos(\theta), \quad n_{y'z} = 0$$

$$n_{z'x} = 0, \quad n_{z'y} = 0, \quad n_{z'z} = 1$$

$$\mathbf{N} = \begin{bmatrix} n_{x'} & n_{y'} & n_{z'} \end{bmatrix} = \begin{bmatrix} n_{x'x} & n_{y'x} & 0 \\ n_{x'y} & n_{y'y} & 0 \\ 0 & 0 & n_{z'z} = 1 \end{bmatrix}$$

Transformation of Stress Components

- Transformation Tensor, **N**

$$\mathbf{N} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transformed stress, *S'*

$$\mathbf{S} = \mathbf{N}^{\mathrm{T}}\mathbf{S}\mathbf{N}$$

Example 7

Body Forces

- **Body Forces** are forces that act on every element of a material and hence on the entire volume of the material.
- Example: Gravitational Forces

$$\mathbf{f}^{(b)} = \int_V \rho \mathbf{b} \, dV = m \int_V \mathbf{b} \, dV = m\mathbf{g}$$

$\mathbf{f}^{(b)}$ – Force Vector

$\rho(\mathbf{x}, t)$ – Density

$\mathbf{b}(\mathbf{x}, t)$ – Body Force Vector

$\mathbf{x}$ – Current Configuration

Surface Forces

- **Surface forces** act on the surface of a material. This surface may be either a part or the whole of the boundary surface

$$\mathbf{f}^{(s)} = \int_S \mathbf{t}\, dS$$

$$\mathbf{f}^{(b)} - \text{Surface Force Vector}$$

$$\mathbf{t}(\mathbf{x}, t) - \text{ Traction Vector}$$

$$dS - \text{ Surface Increment}$$

Equilibrium

- When we speak of Equilibrium, we refer to the Newton's laws of motion and equilibrium given as,

$$\boxed{\sum \mathbf{F} = m\mathbf{a} \quad or \quad \sum \mathbf{F}^s + \sum \mathbf{F}^b = m\mathbf{a}}$$

where $\sum \mathbf{F}$ represents the sum of all Forces (both surface and body), m is mass, and **a** is acceleration.

- Equilibrium ONLY EXISTS if the left hand side (LHS) and right hand side (RHS) of the equation ARE EQUAL.
- In the case where a=0 and v≥0 this equation becomes

$$\boxed{\sum \mathbf{F} = 0 \quad or \quad \sum \mathbf{F}^s + \sum \mathbf{F}^b = 0}$$

- This condition is called "Static Equilibrium".

Derive!

Balance of Linear Momentum

$$\boxed{\begin{aligned}
\frac{d\sigma_{11}}{dx_1} + \frac{d\sigma_{21}}{dx_2} + \frac{d\sigma_{31}}{dx_3} + f_1^b &= 0 \\
\frac{d\sigma_{12}}{dx_1} + \frac{d\sigma_{22}}{dx_2} + \frac{d\sigma_{32}}{dx_3} + f_2^b &= 0 \\
\frac{d\sigma_{13}}{dx_1} + \frac{d\sigma_{23}}{dx_2} + \frac{d\sigma_{33}}{dx_3} + f_3^b &= 0
\end{aligned}}$$

$$\boxed{\frac{d\sigma_{ij}}{dx_j} + f_i^b = \rho a_i}$$

Motion

$$\boxed{\frac{d\sigma_{ij}}{dx_j} + f_i^b = 0}$$

Static Equilibrium

Balance of Angular Momentum

$$\boxed{\sum M = \int_V \mathbf{x}\times\left(\rho\mathbf{v}\right)\, dV = I\boldsymbol{\omega}}$$

Motion

$$\boxed{\sum M = 0}$$

Static Equilibrium

- Proves,

$$\boxed{\sigma_{ij} = \sigma_{ji}}$$

Contact Information

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