Lecture 4

Compatibility Problem

Consider a deformed body with strains equal to the following

Given, $\varepsilon_{xx} = \frac{\delta u}{\delta x} = y$ ①

$\gamma_{xy} = \frac{\delta u}{\delta y} + \frac{\delta v}{\delta x} = xy$ ②

$\varepsilon_{yy} = \frac{\delta v}{\delta y} = y$ ③

Find the Displacements u & v

Integrate ① & ③ to find u, v respectively,

take ① $\frac{\delta u}{\delta x} = y$

$\delta u = y \, \delta x$

$\int$ " $= \int$ "

"the +C"

$\boxed{u = yx + f(y)}$ (I)

take ③ $\frac{\delta v}{\delta y} = y$

$\delta v = y \, \delta y$

$\int$ " $= \int$ "

"+C"

$\boxed{v = \frac{1}{2} y^2 + g(x)}$ (II)

Alternatively, take v & solve ② for u,

take ② $\gamma_{xy} = \frac{\delta u}{\delta y} + \frac{\delta v}{\delta x} = xy$

do $\frac{\delta v}{\delta x} \longrightarrow \frac{\delta}{\delta x}\left[\frac{1}{2} y^2 + g(x)\right] = 0 + \frac{\delta g(x)}{\delta x}$

$\frac{\delta u}{\delta y} + \frac{\delta g(x)}{\delta x} = xy$ $\quad g'(x) = \frac{\delta g(x)}{\delta x}$

$\frac{\delta u}{\delta y} + g'(x) = xy$

$\frac{\delta u}{\delta y} = xy - g'(x)$

$$\frac{\delta u}{\delta y} = xy - g'(x)$$

$$\delta u = \left[xy - g'(x)\right]\delta y$$

$$\int \text{''} = \int \text{''}$$

"+ c" term

$$\boxed{u = \frac{1}{2}xy^2 - y\,g'(x) + h(x)} \quad \text{III}$$

together,

$$\boxed{u = yx + f(y)} \quad \text{I}$$

do not match!

Not compatible!

$$\boxed{v = \frac{1}{2}y^2 + g(x)} \quad \text{II}$$

$$\boxed{u = \frac{1}{2}xy^2 - y\,g'(x) + h(x)} \quad \text{III}$$

How to enforce compatibility!?

$$\gamma_{xy} = \frac{\delta u}{\delta y} + \frac{\delta v}{\delta x}$$

take $\frac{\delta}{\delta x}\frac{\delta}{\delta y}$ of γ_{xy}

$$\frac{\delta^2 \gamma_{xy}}{\delta x \delta y} = \frac{\delta^2}{\delta x \delta y}\left(\frac{\delta u}{\delta y}\right) + \frac{\delta^2}{\delta x \delta y}\left(\frac{\delta v}{\delta x}\right)$$

Assume continuous function,

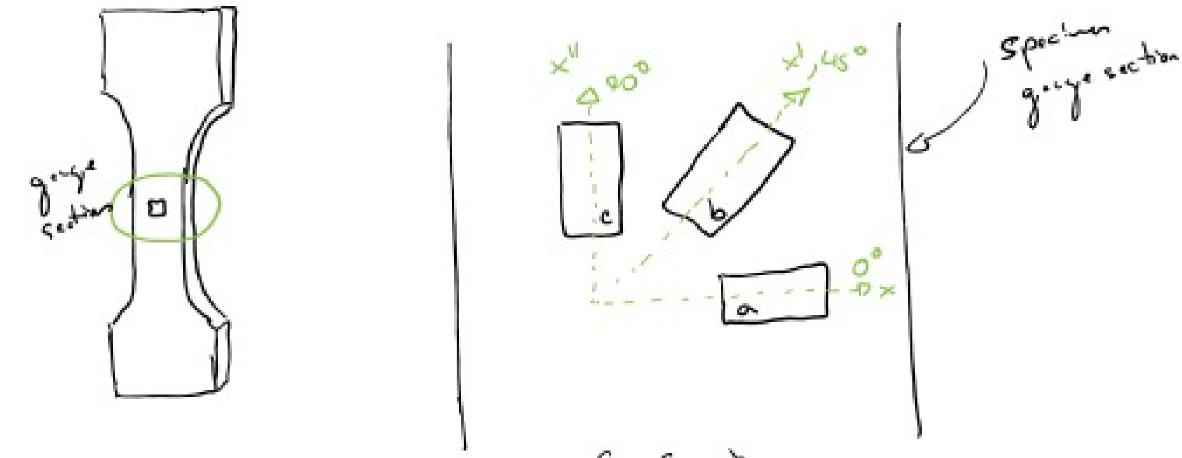
$$\frac{\delta^2 \gamma_{xy}}{\delta x \delta y} = \frac{\delta^2}{\delta y^2}\left(\frac{\delta u}{\delta x}\right) + \frac{\delta^2}{\delta x^2}\left(\frac{\delta v}{\delta y}\right)$$

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial y^2}\left(\varepsilon_{xx}\right) + \frac{\partial^2}{\partial x^2}\left(\varepsilon_{yy}\right)$$

$$\boxed{\frac{\partial^2 \gamma_{xy}}{\partial x \, \partial y} = \frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2}}$$

Compatibility Constraint!!

Example Strain Rosette



Find the state of strain ($\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}$) on this specimen given,

$$\varepsilon_a, \varepsilon_b, \varepsilon_c$$

Lets use the Strain Transformation Eqns!

Use the first Eqn

$$\varepsilon_{x'x'} = \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} + \frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\cdot\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta$$

$\underline{x}$, $\theta = 0°$

$$\varepsilon_a = \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} + \frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\cdot\overset{1}{\cancel{\cos(0)}} + \frac{1}{2}\gamma_{xy}\overset{0}{\cancel{\sin(0)}}$$

$$\varepsilon_a = \varepsilon_{xx}$$

$\underline{x''}$, $\theta = 90°$

$$\varepsilon_c = \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} + \frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\cdot\overset{(-1)}{\cancel{\cos(180°)}} + \frac{1}{2}\gamma_{xy}\overset{0}{\cancel{\sin(180°)}}$$

$$\varepsilon_c = \varepsilon_{yy}$$

x', $\theta = 45^\circ$

$$\epsilon_b = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} + \frac{\varepsilon_{xx} - \varepsilon_{yy}}{2} \cdot \overset{0}{\cancel{\cos(90^\circ)}} + \frac{1}{2}\gamma_{xy} \overset{1}{\cancel{\sin(90^\circ)}}$$

$$\epsilon_b = \frac{1}{2}\left(\varepsilon_{xx} + \epsilon_{yy} + \gamma_{xy}\right)$$

Together,

$$\boxed{\begin{aligned} \epsilon_a &= \epsilon_{xx} \\ \epsilon_b &= \frac{1}{2}\left(\varepsilon_{xx} + \epsilon_{yy} + \gamma_{xy}\right) \\ \epsilon_c &= \epsilon_{yy} \end{aligned}}$$

Example (Boresi)

A machine part in the form of a parallelopiped that is deformed into a given shape. The displacement is given by the following equations.

$$\underline{u} = u\underline{e}_1 + v\underline{e}_2 + w\underline{e}_3$$

$$u = C_1 \cdot xyz$$

$$v = C_2 \cdot xyz$$

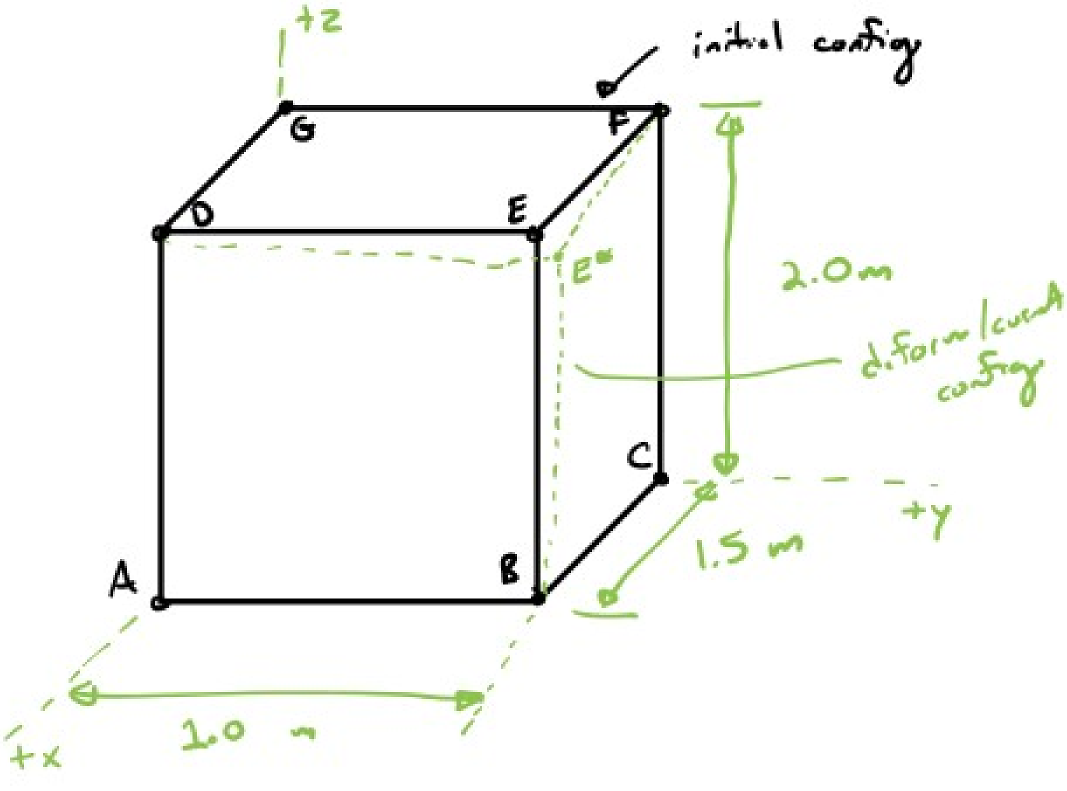
$$w = C_3\, xyz$$

C_1, C_2, C_3 are constants

&

x, y, z are the position within the body

a) Determine the state of strain at point E^* for $x^*, y^*, z^* = (1.504, 1.002, 1.996)$



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- ▷ x, y, z dimensions
- ▷ u, v, w displacement equations
- ▷ x^*, y^*, z^* new dimensions
- ▷ State of Strain $\underline{\underline{\varepsilon}}$

State of strain Eqn

$$\epsilon_{xx} = \frac{\partial u}{\partial x}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

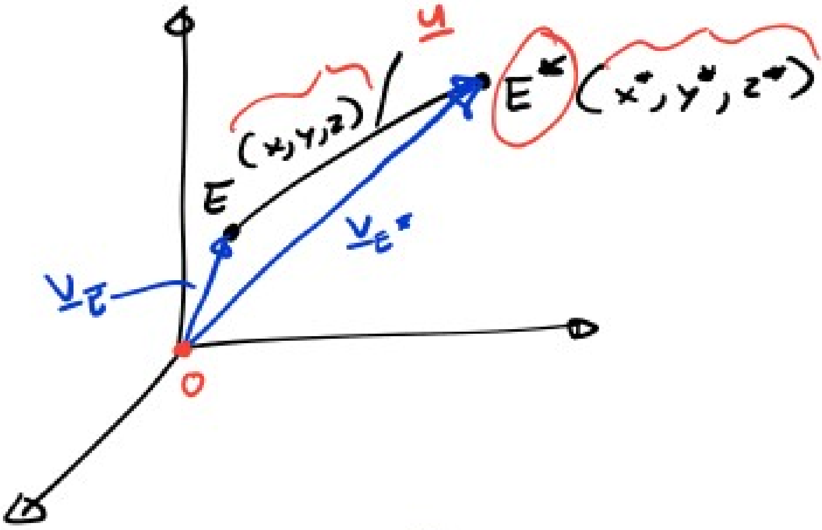
$$\epsilon_{zz} = \frac{\partial w}{\partial z}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$u = C_1 xyz$, $v = C_2 xyz$, $w = C_3 xyz$

C_1, C_2, C_3 are unknowns!

u, v, w are unknowns!

Find u,v,w



$$\underline{u} = \underline{v}_{E^*} - \underline{v}_E$$

$$= u\,\underline{e}_1 + v\,\underline{e}_2 + w\,\underline{e}_3$$

$$\underline{u} = \underline{v}_{E^*} - \underline{v}_E = u\,\underline{e}_1 + v\,\underline{e}_2 + \underline{w}\,\underline{e}_3$$

$$(x^* - x)\,\underline{e}_1 + (y^* - y)\,\underline{e}_2 + (z^* - z)\,\underline{e}_3$$

$$u = x^* - x = (1.504 - 1.5)\,m = \boxed{0.004\ m}$$

$$v = y^* - y = (1.002 - 1)\,m = \boxed{0.002\ m}$$

$$w = z^* - z = (1.996 - 2)\,m = \boxed{-0.004\ m}$$

Find C_1, C_2, C_3

$$\underbrace{0.004}_{\text{Deform}} = \underset{\text{constant}}{C_1}\,\underbrace{xyz}_{\text{initial config}} \qquad 0.002 = C_2\,xyz \qquad -0.004 = C_3\,xyz$$

$$xyz = (1)(1.5)(2) = 3$$

$$C_1 = \frac{0.004}{xyz} = \frac{0.004}{3} \rightarrow u = \frac{0.004}{3}xyz$$

$$C_2 = \frac{0.002}{3} \rightarrow v = \frac{0.002}{3}xyz$$

$$C_3 = \frac{-0.004}{3} \rightarrow w = \frac{-0.004}{3}xyz$$

Solve for $\underline{\underline{\epsilon}}$

$$\epsilon_{xx} = \frac{\delta u}{\delta x} = \frac{0.004}{3}yz = \frac{0.004}{3}(1)(2)$$

E

$$\epsilon_{xx} = 0.00267,$$

repeat for ϵ_{yy}, ϵ_{zz}, γ_{xy}, γ_{yz}, γ_{zx}

initial config

$$\epsilon_{yy} = \frac{\partial v}{\partial y} = \frac{0.002}{3} xz = \boxed{0.002}$$

initial config

$$\epsilon_{zz} = \frac{\partial w}{\partial z} = \frac{-0.004}{3} xy = \boxed{-0.002}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \left(\frac{0.002}{3} yz + \frac{0.004}{3} xz\right)$$

$$\boxed{\gamma_{xy} = 0.00533}$$

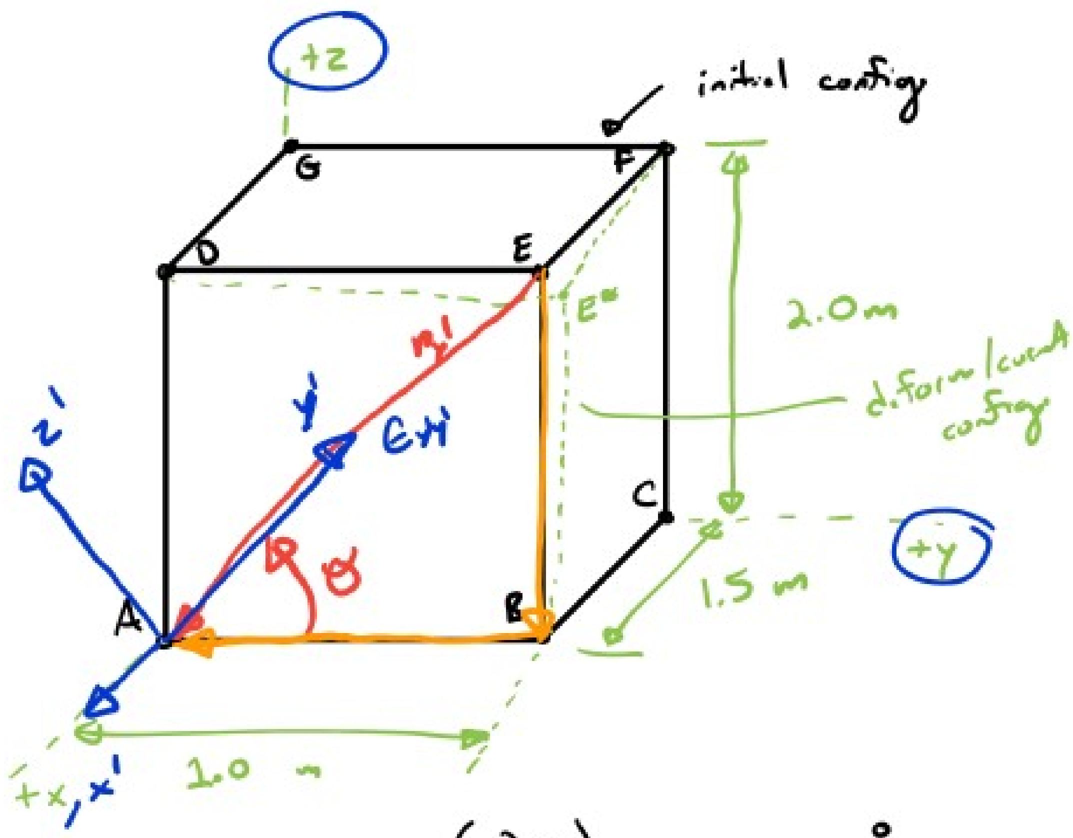
$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \left(-\frac{0.004}{3} xz + \frac{0.002}{3} xy\right)$$

$$\boxed{\gamma_{yz} = -0.003}$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \left(\frac{-0.004}{3} yz + \frac{0.004}{3} xy\right)$$

$$\boxed{\gamma_{zx} = -0.003}$$

b) Determine the normal strain at E in the direction of line EA.



$$\theta = \tan^{-1}\left(\frac{2m}{1m}\right) = 63.435^\circ$$

$$\underline{n}_{y'} = \cos\alpha\, \underline{e}_1 + \cos\beta\, \underline{e}_2 + \cos\gamma\, \underline{e}_3$$

$n_{y'x} = \cos(90^\circ)$ $\quad n_{y'y} = -\cos\theta$ $\quad n_{y'z} = -\sin\theta$

$n_{y'x} = 0$ $\quad n_{y'y} = -0.447$ $\quad n_{y'z} = -0.894$

Find the Normal Strain, $\epsilon_{y'y'}$

$$\epsilon_{y'y'} = \cancel{n_{y'x}^2 \epsilon_{xx}}^{\,0} + n_{y'y}^2 \epsilon_{yy} + n_{y'z}^2 \epsilon_{zz}$$

$$+ \cancel{n_{y'x} n_{y'y} \gamma_{xy}}^{\,0} + n_{y'y} n_{y'z} \gamma_{yz} + \cancel{n_{y'z} n_{y'x} \gamma_{zx}}^{\,0}$$

$$\epsilon_{y'y'} = n_{y'y}^2 \epsilon_{yy} + n_{y'z}^2 \epsilon_{zz} + n_{y'y} n_{y'z} \gamma_{yz}$$

$$= (-0.447)^2 (0.002) + (-0.894)^2 (-0.002)$$

$$+ (-0.447)(-0.894)(-0.002)$$

$$\boxed{\epsilon_{y'y'} = -0.00240}$$

Example Problem 3-14 in Solecki